Original Report

Quasiperiodic Solutions of Quasiperiodic Differential Difference Equations

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Abstract

An existence theorem is given in the paper concerning a quasiperiodic system of differential differerence equations. It says that one can always assure the existence of a quasiperiodic solution by checking several conditions on an obtained approximate solution and further gives a method to obtain an error bound of the approximate solution.

§ 0 Introduction

M. Urabe [5] studied a system of quasiperiodic differential equations and proved an existence theorem of quasiperiodic solutions with the same periods. It says that one can always assure the existence of an exact quasiperiodic solution by checking several conditions on an obtained approximate solution and further gives a method to obtain an error bound of the approximate solution. In order to investigate the properties of quasiperiodic functions, he introduced a notion of pseudoperiodic functions. Therefore his existence theorem had some additional assumptions concerning pseudoperiodic functions. Later he gave a final existence theorem in his paper [6] but did not write its proof yet.

We study a system of quasiperiodic differential difference equations

$$dx(t)/dt = F(t, x(t), x(t+\tau)) \qquad (0.1)$$

and prove an analoguous existence theorem concerning quasiperiodic solutions with the same periods without using any notion of pseudoperiodic functions. Our theorem contains the final theorem by M. Urabe as a special case.

In our previous paper [7] we gave two examples for our main existence theorem suggested by T. Mitsui [2] and Y. Shinohara, A. Kohda and T. Mitsui [4]. Approximate solutions were constructed by the method of Galerkin procedure based on trigonometric polynomials. We established the existence of quasiperiodic solutions to quasiperiodic differential difference equations of 2nd order and the error bounds for the approximate solutions by applying the special case of our existence theorem. We, however, did not write its proof. In this paper we give a complete proof of the existence theorem in the general form.

A function $f(t)$ in t on the real line is called to be quasiperiodic with periods $\omega_1, \dots, \omega_m$ if it is represented as $f(t) = f_0(t, \dots, t)$ for some function $f_0(u_1, \dots, u_m)$ continuous and periodic in each u_j with period ω_j $(j = 1, \dots, m)$. Without any loss of generality we assume that $\omega_j > 0 (j = 1, \dots, m)$ and that the reciprocals of these periods are rationally independent. A function $f(t)$ is said to be almost periodic if from every sequence $\{\alpha_n\}$ one can extract a subsequence $\{\alpha_n'\}$ such that the sequence $\{f(t+\alpha_n')\}$ is uniformly convergent on the real line. It is seen later that a quasiperiodic function is almost periodic. We assume that all functions considered in this paper are continuous on the real line unless we make any note of the matter.

In § 1 we state some facts on almost periodic functions and almost periodic linear differential equations. In § 2 we study some properties of quasiperiodic functions and quasiperiopdic linear differential equations in order to use them in the

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next section. In § 3 we prove the main theorem on quasiperiodic solutions of quasiperiodic differential difference equations.

§ 1 Almost Periodic Linear Systems

It is known [1] that a limit value

$$a(f, \lambda) = \lim_{T \to +\infty} (1/T) \int_0^T f(t) e^{-i\lambda t} dt \qquad (1.1)$$

exists for any almost periodic function $f(t)$ and any real number λ and that there exists a countable set of real numbers Λ such that $a(f, \lambda) = 0$ if $\lambda \in \Lambda$. Denote by the module of f, Mod(f) the smallest additive group of real numbers containing the set Λ for which $a(f, \lambda) \neq 0$.

A system of differential equations

$$dx/dt = A(t)x \qquad (1.2)$$

is called to satisfy an exponential dichotomy if there exists a projection P and positive constants σ_1, σ_2, K_1, and K_2 so that

$$|X(t)PX^{-1}(s)| \leqq K_1 \exp[-\sigma_1(t-s)] \quad \text{for } t \leqq s,$$
$$|X(t)(E-P)X^{-1}(s)| \leqq K_2 \exp[-\sigma_2(s-t)] \quad \text{for } t \leqq s \qquad (1.3)$$

for the fundamental matrix $X(t)$ of the system (1.1) satisfying $X(0) = E$, where E is the unit matrix. Here we introduce any norm $|\ |$ in Euclidean space and denote that $\|f\| = \sup|f(t)|$ for any bounded function $f = f(t)$. We make use of the theorem as follows in the next section.

THEOREM 1.1 [1] Let $A(t)$ be an almost periodic square matrix. Suppose that the system (1.2) satisfies an exponential dichotomy (1.3) and that $f(t)$ is an almost periodic function. Then there is a unique almost periodic solution $\phi(t)$ of of the nonhomogeneous system of differential equations

$$dx/dt = A(t)x + f(t) \qquad (1.4)$$

and Mod(ϕ) $\subset$ Mod(A, f). Furthermore

$$\|\phi\| \leqq (K_1/\sigma_1 + K_2/\sigma_2)\|f\|. \qquad (1.5)$$

where σ_1, σ_2, K_1, and K_2 are constants in (1.3).

In fact, the unique solution $\phi(t)$ is represented as follows

$$\phi(t) = \int_{-\infty}^{+\infty} G(t, s) f(s) ds \qquad (1.6)$$

where

$$G(t, s) = \begin{cases} X(t)PX^{-1}(s) & \text{for } t \geqq s \\ X(t)(P-E)X^{-1}(s) & \text{for } t \leqq s \end{cases} \qquad (1.7)$$

$G(t, s)$ is a piecewise continuous function on the (t, s) plane which is called Green function.

§ 2 Quasiperiodic Linear System

Using a theorem proved by F. Nakajima [3] on the relationships between quasiperiodic functions and almost periodic ones, we prove the closedness of the space of quasiperiodic functions with periods ω_1, $\cdots$, ω_m in the topology of uniform convergence.

THEOREM 2.1 [3] A function $f(t)$ is quasiperiodic with periods $\omega_1, \cdots, \omega_m$ if and only if it is almost periodic and its module has an integral base, namely any $\lambda \in$ Mod(f) is represented as $\lambda = 2\pi(n_1/\omega_1 + \cdots + n_m/\omega_m)$ for some integers $n_1, \cdots, n_m$.

THEOREM 2.2 If a sequence $\{f_n(t)\}$ of quasiperiodic functions with common periods $\omega_1, \cdots, \omega_m$ is uniformly convergent to a function $f(t)$ on the real line, then the limit function $f(t)$ is also quasiperiodic with the same periods $\omega_1, \cdots, \omega_m$.

We shall give a proof of the above theorem. Since each function $f_n(t)$ is almost periodic, it is well known [1] that the limit function $f(t)$ to which the sequence $\{f_n(t)\}$ converges uniformly is also almost periodic. We choose any λ such that $a(f, \lambda)$ defined in (1.1) does not vanish. Set $\varepsilon = |a(f, \lambda)|/4 > 0$. For such $\varepsilon > 0$ there exists a real number T_0 such that

$$|(1/T) \int_0^T f(t) e^{-i\lambda t} dt - a(f, \lambda)| < \varepsilon$$

for any $T \geqq T_0$. On the other hand, by using uniform convergence of the sequence $\{f_n(t)\}$ We can choose a positive integer $N = N(\varepsilon)$ such that $|f_N(t) - f(t)| < \varepsilon$ for any t on the real line. Then

$$|(1/T) \int_0^T f_N(t) e^{-i\lambda t} dt - (1/T) \int_0^T f(t) e^{-i\lambda t} dt|$$
$$\leqq (1/T) \int_0^T |f_N(t) - f(t)|\,|e^{-i\lambda t}| dt$$
$$\leqq (1/T) \int_0^T \varepsilon dt = \varepsilon$$

For the almost periodic function $f_N(t)$ and the value $a(f_N, \lambda)$ we can choose $T_1 \geqq T_0$ such that for any $T \geqq T_1$

$$|(1/T) \int_0^T f_N(t) e^{-i\lambda t} dt - a(f_N, \lambda)| < \varepsilon.$$

It follows that for any $T \geqq T_1$

$$|a(f_N, \lambda)| \geqq |a(f, \lambda)|$$

$$-|a(f,\lambda)-(1/T)\int_0^T f(t)e^{-i\lambda t}dt|$$

$$-|(1/T)\int_0^T f_N(t)e^{-i\lambda t}dt-a(f_N,\lambda)|$$

$$-|(1/T)\int_0^T f_N(t)e^{-i\lambda t}dt$$

$$-(1/T)\int_0^T f(t)e^{-i\lambda t}dt|$$

$$>4\varepsilon-\varepsilon-\varepsilon-\varepsilon=\varepsilon>0$$

Thus we obtain $a(f_N, \lambda)\neq 0$. This implies that $\lambda\in \mathrm{Mod}(f_N)$. By THEOREM 2.1, it follows that $\lambda = 2\pi(n_1/\omega_1+\cdots+n_m/\omega_m)$ for some integers $n_1, \cdots, n_m$ since $f_N(t)$ is quasiperiodic with periods $\omega_1, \cdots, \omega_m$. Then it is concluded that Mod(f) has a finite integral base $2\pi/\omega_1, \cdots, 2\pi/\omega_m$. This implies by THEOREM 2.1 that the function $f(t)$ is quasiperiodic with periods $\omega_1, \cdots, \omega_m$. This completes the proof of THEOREM 2.2.

We obtain a theorem on the existence of a quasiperiodic solution to a linear differential system. This theorem was proved by M. Urabe [3] using the notion of pseudoperiodic functions. We give another proof due to THEOREM 1.1 and THEOREM 2.1 without using any notion of pseudoperiodic functions.

THEOREM 2.3[5] [6] Let $A(t)$ be a quasiperiodic square matrix with periods $\omega_1, \cdots, \omega_m$. Suppose that the system (1.2) satisfies an exponential dichotomy (1.3). Then for any quasiperiodic function $f(t)$ with periods $\omega_1, \cdots, \omega_m$ the nonhomogeneous system (1.4) has a unique quasiperiodic solution $\phi(t)$ with the same periods $\omega_1, \cdots, \omega_m$ given by (1.6), where the Green function $G(t, s)$ is defined in the form (1.7). Moreover the solution $\phi(t)$ satisfies the relation (1.5).

In fact, all assumptions of THEOREM 1.1 are fulfilled since quasiperiodic functions are almost periodic. In order to complete the proof of THEOREM 2.3, it is sufficient by THEOREM 1.1 to prove that the unique almost periodic solution $\phi(t)$ is quasiperiodic with periods $\omega_1, \cdots, \omega_m$. If $\lambda\in \mathrm{Mod}(\phi)$, it follows from the conclusion of THEOREM 1.1 that $\lambda\in \mathrm{Mod}(A, f)$. By THEOREM 2.1 it can be represented as $\lambda=2\pi(n_1/\omega_1+\cdots+n_m/\omega_m)$ for some integers $n_1, \cdots, n_m$. Thus it is concluded that $\phi(t)$ is quasiperiodic with period $\omega_1, \cdots, \omega_m$.

§ 3 Quasiperiodic Systems of Differential Difference Equations

We are in a position to obtain our main theorem concerning a system of differential difference equations (0.1) defined on the real line, where τ is a constant.

THEOREM 3.1 Let D be a bounded domain in Euclidean space with a norm $|\ |$. Assume that the given function $F(t, x, y)$ in (0.1) is quasiperiodic with periods $\omega_1, ..., \omega_m$ in t on the real line and continuously differentiable in (x, y) on the domain $D\times D$. Suppose that the system of differential difference equations (0.1) has an approximate solution $x=\bar{x}(t)$ quasiperiodic with the same periods $\omega_1, ..., \omega_m$ lying in D for any t and satisfying

$$|d\bar{x}(t)/dt-F(t, \bar{x}(t), \bar{x}(t+\tau))|\leqq r \qquad (3.1)$$

for all t. Further suppose that there exist a positive constant δ, nonnegative constants $\varkappa$ and μ and a square matrix $A(t)$ quasiperiodic with the same periods $\omega_1, ..., \omega_m$ satisfying the conditions as follows:

(i) The linear system (1.2) satisfies an exponential dichotomy (1.3),

(ii) $D_\delta=\{x: |x-\bar{x}(t)|\leqq\delta \text{ for some } t\}\subset D$,

(iii) The relations $|\phi(t, x, y)-A(t)|\leqq\varkappa/M$ and $|\Psi(t, x, y)|\leqq\mu$ hold for any t on the real line and any (x, y) in $D_\delta\times D_\delta$,

(iv) The relations $\varkappa+M\mu<1$ and $Mr/(1-\varkappa-M\mu)\leqq\delta$ hold.

Here $\Phi(t, x, y)$ and $\Psi(t, x, y)$ are Jacobian matrices of the function $F(t, x, y)$ with respect to x and y respectively and $M=K_1/\sigma_1+K_2/\sigma_2$, where σ_1, σ_2, K_1 and K_2 are constants in (1.3). Then the given system (0.1) has a unique solution $x=x(t)$ in t on the real line quasiperiodic with the same periods $\omega_1, ..., \omega_m$ lying in D_δ for any t. Moreover it satisfies the relation

$$|x(t)-\bar{x}(t)|\leqq Mr/(1-\varkappa-M\mu) \qquad (3.2)$$

for all t on the real line.

We shall give a proof of this theorem. For the given approximate solution $x=\bar{x}(t)$ we denote that

$$h(t)=d\bar{x}(t)/dt-F(t, \bar{x}(t), \bar{x}(t+\tau)) \qquad (3.3)$$

It follows form (3.1) that $|h(t)|\leqq r$ for any t.

Rewriting the relation (3.3), we have

$$d\bar{x}(t)/dt = A(t)\bar{x}(t)+[F(t, \bar{x}(t), \bar{x}(t+\tau)) -A(t)\bar{x}(t)+h(t)] \quad (3.4)$$

Noting that the nonhomogeneous term of the above system (3.4) is quasiperiodic with periods $\omega_1, \cdots, \omega_m$, we apply THEOREM 2.3 to the system (3.4) and obtain

$$\bar{x}(t) = \int_{-\infty}^{+\infty} G(t, s)[F(s, \bar{x}(s), \bar{x}(s+\tau)) -A(s)\bar{x}(s)+h(s)]ds \quad (3.5)$$

where $G(t, s)$ is the Green function (1.7).

To seek an exact quasiperiodic solution to the system (0.1), we define an iterative process of the form:

$$x_0(t) = \bar{x}(t) \quad (3.6)$$

$$x_{n+1}(t) = \int_{-\infty}^{+\infty} G(t, s)[F(s, x_n(s), x_n(s+\tau)) -A(s)x_n(s)]ds \quad (3.7)$$

for $n = 0, 1, \cdots$. We shall prove by induction that this process can be continued infinitely in the space of quasiperiodic functions with periods $\omega_1, \cdots, \omega_m$ and that the inequalities

$$\|x_{n+1}-x_n\| \leqq (\varkappa+M\mu)^n\|x_1-x_0\| \quad (3.8)$$

and

$$\|x_{n+1}-\bar{x}\| \leqq \delta \quad (3.9)$$

hold for $n = 0, 1, \cdots$.

In fact, the inequality (3.8) is evident for $n = 0$. It follows from (3.5), (3.6) and (3.7) that

$$x_1(t)-x_0(t) = -\int_{-\infty}^{+\infty} G(t, s)h(s)ds$$

This implies from (1.5) and the conditon (iv) that

$$\|x_1-\bar{x}\| = \|x_1-x_0\| \leqq (K_1/\sigma_1+K_2/\sigma_2)\|h\| = M\|h\| \leqq Mr \leqq (1-\varkappa-M\mu)\delta < \delta$$

This proves (3.9) for $n = 0$. To prove our statement by induction, let us assume that the iterative functions $x_n(t)$ have been well defined and satisfy (3.8) and (3.9) up to $n-1$. For n we can make $x_{n+1}(t)$ by (3.7) and the condition (ii). It follows from (3.7) that

$$\begin{aligned} &x_{n+1}(t)-x_n(t) \\ &= \int_{-\infty}^{+\infty} G(t, s)\{F(s, x_n(s), x_n(s+\tau)) \\ &\quad -F(s, x_{n-1}(s), x_{n-1}(s+\tau)) \\ &\quad -A(s)[x_n(s)-x_{n-1}(s)]\}ds \\ &= \int_{-\infty}^{+\infty} G(t, s) \int_0^1 \{[\Phi(s, x_n^\theta(s), x_n^\theta(s+\tau)) \\ &\quad -A(s)][x_n(s)-x_{n-1}(s)] \\ &\quad +\Psi(s, x_n^\theta(s), x_n^\theta(s+\tau)) \\ &\quad [x_n(s+\tau)-x_{n-1}(s+\tau)]\}d\theta ds, \end{aligned} \quad (3.10)$$

Where $x_n^\theta(s) = x_{n-1}(s)+\theta[x_n(s)-x_{n-1}(s)\}, 0 \leqq \theta \leqq 1$. Since $x_n(t)$ and $x_{n-1}(t)$ lie in the domain D_δ for any t by the assumptions of induction, $x_n^\theta(t)$ also lies in the domain D_δ and then in the domain D by the condition (ii). It follows from the condition (iii) and the conclusion (1.5) of THEOREM 2.3 that

$$\begin{aligned} \|x_{n+1}-x_n\| &\leqq (K_1/\sigma_1+K_2/\sigma_2)(\varkappa/M+\mu)\|x_n-x_{n-1}\| \\ &\leqq (\varkappa+M\mu)\|x_n-x_{n-1}\| \\ &\leqq (\varkappa+M\mu)^n\|x_1-x_0\| \end{aligned} \quad (3.11)$$

This implies the relation (3.8). Moreover it follows that

$$\begin{aligned} \|x_{n+1}-\bar{x}\| &= \sum_{k=0}^{n} (\varkappa+M\mu)^k\|x_1-x_0\| \\ &\leqq (1-\varkappa-M\mu)\|x_1-x_0\| \\ &\leqq Mr/(1-\varkappa-M\mu) < \delta \end{aligned} \quad (3.12)$$

This proves the relation (3.9).

Then we obtain an infinite sequence $\{x_n(t)\}$ in the space of quasiperiodic functions with periods $\omega_1, \cdots, \omega_m$ by the iterative process (3.6) and (3.7). It is easy to see from (3.8) and the condition (iv) that the sequence $\{x_n(t)\}$ is uniformly convergent to a function $x(t)$ on the real line. By THEOREM 2.2 it can be concluded that the function $x(t)$ is quasiperiodic with periods $\omega_1, \cdots, \omega_m$ to the system (0.1) lying in D_δ and (3.7), we have (3.2) and

$$x(t) = \int_{-\infty}^{+\infty} G(t, s)[F(s, x(s), x(s+\tau)) -A(s)x(s)]ds$$

respectively. The latter implies that

$$\begin{aligned} dx(t)/dt &= A(t)x(t) \\ &\quad +[F(t, x(t), x(t+\tau))-A(t)x(t)] \\ &= F(t, x(t), x(t+\tau)) \end{aligned}$$

Hence $x(t)$ is our desired solution quasiperiodic with periods $\omega_1, \cdots, \omega_m$ to the system (0.1) lying in D_δ for any t and satisfying (3.2).

In order to prove the uniqueness of quasiperiodic solutions with periods $\omega_1, \cdots, \omega_m$ to the system (0.1) lying in D_δ for any t, we consider another solution $y(t)$ with the same properties. Then we have

$$\begin{aligned} dy(t)/dt &= F(t, y(t), y(t+\tau)) \\ &= A(t)y(t)+[F(t, y(t), y(t+\tau)) \\ &\quad -A(t)y(t)] \end{aligned}$$

and hence

$$y(t) = \int_{-\infty}^{+\infty} G(t, s)[F(s, y(s), y(s+\tau)) - A(s)y(s)]ds$$

Using the relation (1.5) and the condition (iii), we have

$$\|x-y\| \leqq (\kappa + M\mu)\|x-y\|$$

by the same arguments as those in proceeding from (3.10) to (3.11). It follows from the condition (iv) that $\|x-y\| = 0$. This proves the uniqueness of solutions quasiperiodic with periods $\omega_1, \cdots, \omega_m$ to the given system (0.1) lying in D_δ for all t. This completes the proof of THEOREM 3.1.

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