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Asymptotic decay of solutions to the Cauchy problem for the conservation law with Harabetian-type viscosity

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Abstract. In the present paper, we consider the asymptotic decay of solutions to the Cauchy problem for the conservation law with Harabetian-type nonlinear viscosity. We obtain the almost optimal decay properties of solution which tends toward a constant state as time goes to infinity. Important are constructing the time-weighted energy inequalities with the aid of some interpolation inequalities.

要旨：本論文では、Harabetian型の非線形粘性を有する保存則のCauchy問題の解の漸近的減衰について考える。ここではある定数状態に漸近する解の殆ど最良の時間減衰評価を得る。重要なことは幾つかの補間不等式を用いて時間重み付きエネルギー不等式を構成することである。

1. Introduction and main theorems.

We consider the Cauchy problem for the viscous conservation law with Harabetian-type viscosity

$$\begin{cases} \partial_t u + \partial_x \Big(f(u) - \mu\, \partial_x \big((1 + |u|^\gamma)\, u \big) \Big) = 0 \quad \big(t > 0,\ x \in \mathbb{R} \big), \\ u(0,x) = u_0(x) \to 0 \quad \big(x \to \pm\infty \big), \end{cases} \tag{1.1}$$

where $u(t,x)$ is the unknown function of $t > 0$ and $x \in \mathbb{R}$, the so-called conserved quantity,

$$f(u) - \mu\, \partial_x \Big((1 + |u|^\gamma)\, u \Big) \quad \big(\gamma > 0,\ \mu > 0 \big)$$

is the total flux, in particular, the first and second terms are the said to be convective and viscous/diffusive fluxes, respectively, and $u_0(x)$ is the given initial data. In our viscous flux, we note that the function $A(u) = \mu\,(1 + |u|^\gamma)\, u$ satisfies $A'(u) \geq \mu > 0$ for $u \in \mathbb{R}$ and therefore our viscosity can be classified into Harabetian-type (see [2, 28], see also [6, 15, 19], cf. [7, 12, 21-27] (for the Matsumura-Nishihara and its related nonlinear models), [18] (for diffusive dispersive flux)).

Yoshida [28] very recently showed that the solution tends toward a constant state 0 as time goes to infinity with the aid of a technical energy method. This result is precisely stated as follows.

Theorem 1.1 (*Yoshida [28]*). *Let $\mu > 0, \gamma > 0$ and the convective flux $f \in C^1(\mathbb{R})$. Assume the initial data satisfy $u_0 \in H^1$. Then the Cauchy problem* (1.1) *has a unique global in time solution* u *satisfying*

$$
\begin{cases}
u \in C^0\big([0,\infty);H^1\big),\\
|u|^{\gamma}\,\partial_x u \in C^0\big([0,\infty);L^2\big)\cap L^2\big(0,\infty;L^2\big),\\
|u|^{2\gamma}\,\partial_x u \in L^2\big(0,\infty;L^2\big),\\
\partial_t u \in L^2\big(0,\infty;L^2\big),
\end{cases}
$$

and the asymptotic behavior

$$
\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(t,x)| = 0.
$$

We further obtain the decay rate estimates in time of the stability in Theorem1.1 by Hashimoto-Ueda-Kawashima [3]. The main results are stated as follows.

Theorem 1.2 (*Decay Properties I*)**.** *Under the same assumptions in Theorem 1.1, the unique global in time solution u of the Cauchy problem* (1.1) *has the following time-decay estimates*

$$
\begin{cases}
\|u(t)\|_{L^q} \le C_{q,u_0}(1+t)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)},\\
\|u(t)\|_{L^\infty} \le C_{q,u_0}(\epsilon)(1+t)^{-\frac{1}{4}+\epsilon}
\end{cases}
$$

for $q\in[2,\infty)$ and any $\epsilon>0$.

Theorem 1.3 (*Decay Properties II*)**.** *Under the same assumptions in Theorem 1.1, if the initial data further satisfies $u_0\in L^1$, then the unique global in time solution u of the Cauchy problem* (1.1) *has the following time-decay estimates*

$$
\begin{cases}
\|u(t)\|_{L^q} \le C_{q,u_0}(1+t)^{-\frac{1}{2}\left(1-\frac{1}{q}\right)},\\
\|u(t)\|_{L^\infty} \le C_{q,u_0}(\epsilon)(1+t)^{-\frac{1}{2}+\epsilon}
\end{cases}
$$

for $q\in[1,\infty)$ and any $\epsilon>0$.

Remark 1.4. The decay rates in Theorems 1.2 and 1.3 are almost optimal. In fact, they are quite or almost the same as those of viscous conservation laws in Harabetian [2], Matsumura-Nishihara [13] and Yoshida [20, 21, 24, 27].

The proofs of Theorems 1.2 and 1.3 are provided by applying the arguments in Hashimoto-Ueda- Kawashima [3], that is, the time-weighted energy method (see also [3, 20, 21, 24, 27] and so on, cf. [2]).

This paper is organized as follows. In Section 2, we give the uniform energy estimates of solution to the problem (1.1), construct the time-weighted L^q-energy estimates with $2\le q<\infty$ and finally show Theorem 1.2. In Section 3, we also construct the L^1 and time-weighted L^q-energy estimates with $1<q<\infty$, and finally show Thorem 1.3.

Some Notation. We denote by C generic positive constants unless they need to be distinguished. In particular, use $C_{\alpha,\beta,\cdots}$ when we emphasize the dependency on $\alpha,\ \beta,\ \cdots$.

For function spaces, $L^p=L^p(\mathbb{R})$ and $H^k=H^k(\mathbb{R})$ denote the usual Lebesgue space and *k*-th order Sobolev space on the whole space $\mathbb{R}$ with norms $||\cdot||_{L^p}$ and $||\cdot||_{H^k}$, respectively.

2. Decay properties I.

In this Section, we are going to obtain the time-decay estimates of the solution u to the problem (1.1) in Theorem 1.2, that is,

$$\begin{cases} \| u(t) \|_{L^q} \leq C_{q,u_0} (1+t)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}, \\ \| u(t) \|_{L^\infty} \leq C_{q,u_0}(\epsilon)(1+t)^{-\frac{1}{4}+\epsilon} \end{cases}$$

for $q \in [2, \infty)$ and any $\epsilon > 0$. We prepare the uniform energy estimates and some interpolation inequalities. In fact, we can get from [28] the following uniform energy estimates.

Proposition 2.1 (*Uniform Estimates*)**.** *Under the same assumptions in Theorem 1.1, the unique global in time solution* u *of the Cauchy problem* (1.1) *has the following uniform energy estimates*

$$\begin{aligned} &\| u(t) \|_{H^1}^2 + \| u(t) \|_{L^{\gamma+2}}^{\gamma+2} + \int_{-\infty}^{\infty} |u|^{2\gamma} \left|\partial_x u\right|^2 \mathrm{d}x \\ &+ \int_0^t \int_{-\infty}^{\infty} \left(|u|^\gamma + |u|^{2\gamma}\right) \left|\partial_x u\right|^2 \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_{-\infty}^{\infty} |u|^{2\gamma} \left|\partial_t u\right|^2 \mathrm{d}x\mathrm{d}\tau \\ &+ \int_0^t \| \partial_t u(\tau) \|_{L^2}^2 \, \mathrm{d}\tau \leq C_{\mu,\gamma,u_0} \quad \left(t \geq 0\right). \end{aligned}$$

Next, we can get the following interpolation inequalities.

Lemma 2.2. *We have the following interpolation inequalities.*

(1) *For any* $2 \leq q < \infty$*, there exists a positive constant* C_q *such that*

$$\| u(t) \|_{L^q} \leq C_q \left(\int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x \right)^{\frac{2}{q+2}} \left(\int_{-\infty}^{\infty} |u|^{q-2} \left|\partial_x u\right|^2 \mathrm{d}x \right)^{\frac{q-2}{q(q+2)}} \quad \left(t \geq 0\right).$$

(2) *There exists a positive constant* C_q *such that*

$$\| u(t) \|_{L^\infty} \leq C_q \left(\int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{q+2}} \left(\int_{-\infty}^{\infty} |u|^{q-2} \left|\partial_x u\right|^2 \mathrm{d}x \right)^{\frac{1}{q+2}} \quad \left(t \geq 0\right).$$

Proof of Lemma 2.2. Noting

$$|u|^s \leq s \int_{-\infty}^{\infty} |u|^{s-1} \left|\partial_x u\right| \mathrm{d}x \tag{2.1}$$

for $s \geq 1$ and using the Cauchy-Schwarz inequality, we have

$$\| u(t) \|_{L^\infty} \leq s^{\frac{1}{s}} \left(\int_{-\infty}^{\infty} |u|^{2s-q} \, \mathrm{d}x \right)^{\frac{1}{2s}} \left(\int_{-\infty}^{\infty} |u|^{q-2} \left|\partial_x u\right|^2 \mathrm{d}x \right)^{\frac{1}{2s}} \quad \left(t \geq 0\right). \tag{2.2}$$

We choose $s = (q+2)/2$ to the right-hand side of (2.2), and get (2), that is,

$$\| u(t) \|_{L^\infty} \leq \left(\frac{q+2}{2} \right)^{\frac{2}{q+2}} \left(\int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{q+2}} \left(\int_{-\infty}^{\infty} |u|^{q-2} \left|\partial_x u\right|^2 \mathrm{d}x \right)^{\frac{1}{q+2}} \quad \left(t \geq 0\right). \tag{2.3}$$

Further substituting (2.3) into

$$\| u(t) \|_{L^q}^q \leq \| u(t) \|_{L^\infty}^{q-2} \| u(t) \|_{L^2}^2, \tag{2.4}$$

then, (2.4) becomes

$$\| u(t) \|_{L^q}^q \leq \left(\frac{q+2}{2} \right)^{\frac{2(q-2)}{q+2}} \left(\int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x \right)^{\frac{2q}{q+2}} \left(\int_{-\infty}^{\infty} |u|^{q-2} \left|\partial_x u\right|^2 \mathrm{d}x \right)^{\frac{q-2}{q+2}} \quad \left(t \geq 0\right). \tag{2.5}$$

From (2.5), we immediately get (1).

Thus, the proof of Lemma 2.2 is completed.

By using Lemma 2.2, we show the time-weighted L^q-energy estimates to u with $2 \le q < \infty$ stated as the next proposition.

Proposition 2.3 (*Time-weighted L^q-energy Estimates*)**.** *Assume that the same assumptions in Theorem 1.1. For any $q \in [2, \infty)$, there exist positive constants α and $C_{\alpha,q}$ such that the unique global in time solution u of the Cauchy problem* (1.1) *has the following uniform energy estimate*

$$\begin{aligned}
&(1+t)^{\alpha} \| u(t) \|_{L^q}^q + \int_0^t (1+\tau)^{\alpha} \int_{-\infty}^{\infty} |u|^{q-2} |\partial_x u|^2 \, \mathrm{d}x \mathrm{d}\tau \\
&+ \int_0^t (1+\tau)^{\alpha} \int_{-\infty}^{\infty} |u|^{q+\gamma-2} |\partial_x u|^2 \, \mathrm{d}x \mathrm{d}\tau \\
&\le C_{\alpha,q} \| u_0 \|_{L^q}^q + C_{\alpha,q} \int_0^t (1+\tau)^{\alpha - \frac{q+2}{4}} \left(\int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x \right)^{\frac{q}{2}} \mathrm{d}\tau.
\end{aligned}$$

Proof of Proposition 2.3. For $q \ge 2$, multiplying the equation in (1.1) by $|u|^{q-2} u$, and integrating the resultant formula with respect to x, we have, after integration by parts, that

$$\begin{aligned}
&\frac{1}{q} \frac{\mathrm{d}}{\mathrm{d}t} \| u(t) \|_{L^q}^q + \int_{-\infty}^{\infty} \partial_x \left(\int_0^u |s|^{q-2} s f'(s) \, \mathrm{d}s \right) \mathrm{d}x \\
&+ \mu (q-1) \int_{-\infty}^{\infty} |u|^{q-2} |\partial_x u|^2 \, \mathrm{d}x + \mu (q-1)(\gamma+1) \int_{-\infty}^{\infty} |u|^{q+\gamma-2} |\partial_x u|^2 \, \mathrm{d}x = 0.
\end{aligned} \tag{2.6}$$

Further multiplying the equation in (2.6) by $(1+t)^{\alpha}$ for $\alpha > 0$, and integrating the resultant formula with respect to t, we arrive at

$$\begin{aligned}
&\frac{1}{q} (1+t)^{\alpha} \| u(t) \|_{L^q}^q + \mu (q-1) \int_0^t (1+\tau)^{\alpha} \int_{-\infty}^{\infty} |u|^{q-2} |\partial_x u|^2 \, \mathrm{d}x \mathrm{d}\tau \\
&+ \mu (q-1)(\gamma+1) \int_0^t (1+\tau)^{\alpha} \int_{-\infty}^{\infty} |u|^{q+\gamma-2} |\partial_x u|^2 \, \mathrm{d}x \mathrm{d}\tau \\
&= \frac{1}{q} \| u_0 \|_{L^q}^q + \frac{\alpha}{q} \int_0^t (1+\tau)^{\alpha-1} \| u(\tau) \|_{L^q}^q \, \mathrm{d}\tau.
\end{aligned} \tag{2.7}$$

By using Lemma 2.2, the second term on the right-hand side of (2.7) is estimated as

$$\begin{aligned}
&\frac{\alpha}{q} \int_0^t (1+\tau)^{\alpha-1} \| u(\tau) \|_{L^q}^q \, \mathrm{d}\tau \\
&\le \int_0^t (1+\tau)^{\alpha} \left(\int_{-\infty}^{\infty} |u|^{q-2} |\partial_x u|^2 \, \mathrm{d}x \right)^{\frac{q-2}{q+2}} \left(C_{\alpha,q} (1+\tau)^{-\frac{q+2}{4}} \left(\int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x \right)^{\frac{q}{2}} \right)^{\frac{4}{q+2}} \mathrm{d}\tau \\
&\le \epsilon \int_0^t (1+\tau)^{\alpha} \int_{-\infty}^{\infty} |u|^{q-2} |\partial_x u|^2 \, \mathrm{d}x \mathrm{d}\tau + C_{\alpha,q,\epsilon} \int_0^t (1+\tau)^{\alpha - \frac{q+2}{4}} \left(\int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x \right)^{\frac{q}{2}} \mathrm{d}\tau
\end{aligned} \tag{2.8}$$

for any $\epsilon > 0$. Choosing ϵ suitably small, substituting (2.8) into (2.7), we obtain the desired time-weighted L^q -energy estimate to u with $2 \le q < \infty$.

Thus, the proof of Proposition 2.3 is completed.

Finally, in this section, we prove Theorem 1.2.

Proof of Theorem 1.2. By choosing α suitably large and using Proposition 2.1 to the inequality in Proposition 2.3, we obtain the L^q-estimate for u, that is,

$$\| u(t) \|_{L^q} \leq C_{q,u_0} (1+t)^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})} \quad \big(t \geq 0\big) \tag{2.9}$$

for $q \in [2, \infty)$.

We finally show the L^∞-estimate. From (2.9), by using the Gagliardo-Nirenberg inequality (see [1, 16, 17]), that is,

$$\| u(t) \|_{L^\infty} \leq C_{q,\theta} \| u(t) \|_{L^q}^{1-\theta} \| \partial_x u(t) \|_{L^2}^{\theta} \quad \big(t \geq 0\big) \tag{2.10}$$

for any $(q, \theta) \in [1, \infty) \times (0,\, 1]$; $\theta/2 = (\, 1 - \theta\,)/2$. Substituting (2.9) into (2.10), we obtain the L^∞-estimate for u, that is,

$$\| u(t) \|_{L^\infty} \leq C_{q,u_0}(\,\theta\,)\, (1+t)^{-\frac{1}{4}+\theta} \quad \big(t \geq 0\big) \tag{2.11}$$

for any $0 < \theta \leq 1$.

Thus, the proof of Theorem 1.2 is completed.

3. Decay properties II.

In this Section, we further obtain the time-decay estimates of the solution u to the problem (1.1) in Theorem 1.3, that is,

$$\begin{cases} \| u(t) \|_{L^q} \leq C_{q,u_0}(\,\epsilon\,)\, (1+t)^{-\frac{1}{2}(1-\frac{1}{q})}, \\ \| u(t) \|_{L^\infty} \leq C_{q,u_0}(\,\epsilon\,)\, (1+t)^{-\frac{1}{2}+\epsilon} \end{cases}$$

for $q \in [1, \infty)$ and any $\epsilon > 0$ under the condition $u_0 \in H^1 \cap L^1$. We first show the L^1-estimate for u. To do that, we use the Friedrichs mollifier $\rho_\delta *$, where $\rho_\delta(s) := \delta^{-1}\, \rho(\, s/\delta\,)$ with

$$\begin{aligned} &\rho \in C_0^\infty(\mathbb{R}), \quad \rho(s) \geq 0 \quad (\, s \in \mathbb{R}\,), \\ &\mathrm{supp}\{\rho\} \subset \{ s \in \mathbb{R} \mid |\, s\,| \leq 1 \}, \quad \int_{-\infty}^{\infty} \rho(s)\, \mathrm{d}s = 1. \end{aligned}$$

Some useful properties of the mollifier are stated in the next lemma (for the proof, see [3] and so on).

Lemma 3.1. *We have the following properties.*

(1) $\lim\limits_{\delta \to 0} \, (\, \rho_\delta * \mathrm{sgn}\,)\, (s) = \mathrm{sgn}(s) \quad (s \in \mathbb{R}).$

(2) $\lim\limits_{\delta \to 0} \displaystyle\int_0^s (\, \rho_\delta * \mathrm{sgn}\,)\, (\eta)\, \mathrm{d}\eta = |\, s\,| \quad (s \in \mathbb{R}).$

(3) $(\, \rho_\delta * \mathrm{sgn}\,)\Big|_{s=0} = 0.$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}s}\, (\, \rho_\delta * \mathrm{sgn}\,)\, (s) = 2\, \rho_\delta(s) \geq 0 \quad (s \in \mathbb{R}).$

Here

$$(\, \rho_\delta * \mathrm{sgn}\,)\, (s) := \int_{-\infty}^{\infty} \rho_\delta(s-y)\, \mathrm{sgn}(y)\, \mathrm{d}y \quad (s \in \mathbb{R})$$

and

$$\operatorname{sgn}(s) := \begin{cases} -1 & \big(s < 0\big), \\ 0 & \big(s = 0\big), \\ 1 & \big(s > 0\big). \end{cases}$$

By making use of Lemma 3.1, we obtain the following L^1-estimate for u.

Proposition 3.2 (L^1-estimate)**.** *Assume that the same assumptions in Theorem 1.3. The unique global in time solution u of the Cauchy problem* (1.1) *has the following L^1-estimate*

$$\| u(t) \|_{L^1} \le \| u_0 \|_{L^1} \quad \big(t \ge 0\big).$$

Proof of Proposition 3.2. Multiplying the equation in (1.1) by $(\rho_\delta * \operatorname{sgn})(u)$ and integrating the resultant formula with respect to x, we have, after integration by parts, that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} \int_0^u (\rho_\delta * \operatorname{sgn})(s) \, \mathrm{d}s\mathrm{d}x + \int_{-\infty}^{\infty} \partial_x \left(\int_0^u (\rho_\delta * \operatorname{sgn})(s) \, f'(s) \, \mathrm{d}s \right) \mathrm{d}x \\ &+ \mu \int_{-\infty}^{\infty} \frac{\mathrm{d}(\rho_\delta * \operatorname{sgn})}{\mathrm{d}u}(u) \, \partial_x \Big(\big(1 + |u|^\gamma\big) u \Big) \, \mathrm{d}x = 0. \end{aligned} \tag{3.1}$$

Integrating (3.1) with respect to t and using Lemma 3.1, we get as follows.

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_0^{u(t)} (\rho_\delta * \operatorname{sgn})(s) \, \mathrm{d}s\mathrm{d}x + 2\mu \int_0^t \int_{-\infty}^{\infty} \rho_\delta(u) \, \big|\partial_x u\big|^2 \, \mathrm{d}x\mathrm{d}\tau \\ &+ 2\mu(\gamma + 1) \int_0^t \int_{-\infty}^{\infty} \rho_\delta(u) \, |u|^\gamma \, \big|\partial_x u\big|^2 \, \mathrm{d}x\mathrm{d}\tau = \int_{-\infty}^{\infty} \int_0^{u_0} (\rho_\delta * \operatorname{sgn})(s) \, \mathrm{d}s\mathrm{d}x. \end{aligned} \tag{3.2}$$

By using Lemma 3.1, we note

$$\big| (\rho_\delta * \operatorname{sgn})(s) \, \mathrm{d}s \big| \le (\rho_\delta * \operatorname{sgn})(|u|) \, |u| \le |u|, \tag{3.3}$$

and

$$\lim_{\delta \to 0} \int_{-\infty}^{\infty} \int_0^{u(t)} (\rho_\delta * \operatorname{sgn})(s) \, \mathrm{d}s = \| u(t) \|_{L^1}, \tag{3.4}$$

for $t \in [0, \infty)$. Using (3.3) and (3.4), taking the limit $\delta \to 0$ in (3.2) and noting $\rho_\delta \ge 0$ from $\rho \ge 0$, we obtain the desired L^1-estimate to u.

Thus, the proof of Proposition 3.2 is completed.

Similarly in Section 2, we prepare the following interpolation inequalities.

Lemma 3.3. *We have the following interpolation inequalities.*

(1) *For any $1 < q < \infty$, there exists a positive constant C_q such that*

$$\| u(t) \|_{L^q} \le C_q \left(\int_{-\infty}^{\infty} |u| \, \mathrm{d}x \right)^{\frac{2}{q+1}} \left(\int_{-\infty}^{\infty} |u|^{q-2} \big|\partial_x u\big|^2 \, \mathrm{d}x \right)^{\frac{q-1}{q(q+1)}} \quad \big(t \ge 0\big).$$

(2) *There exists a positive constant* C_q *such that*

$$\| u(t) \|_{L^\infty} \le C_q \left(\int_{-\infty}^{\infty} |\, u \,| \, \mathrm{d}x \right)^{\frac{1}{q+1}} \left(\int_{-\infty}^{\infty} |\, u \,|^{q-2} \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x \right)^{\frac{1}{q+1}} \quad \big(t \ge 0\big).$$

Proof of Lemma 3.3. Noting (2.1) in Section 2 and using the Cauchy-Schwarz inequality, we have

$$\| u(t) \|_{L^\infty} \le s^{\frac{1}{s}} \left(\int_{-\infty}^{\infty} |\, u \,|^{2s-q} \, \mathrm{d}x \right)^{\frac{1}{2s}} \left(\int_{-\infty}^{\infty} |\, u \,|^{q-2} \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x \right)^{\frac{1}{2s}} \quad \big(t \ge 0\big). \tag{3.5}$$

We choose $s = (\, q+1 \,)/2$ to the right-hand side of (3.5), and get (2), that is,

$$\| u(t) \|_{L^\infty} \le \left(\frac{q+1}{2} \right)^{\frac{2}{q+1}} \left(\int_{-\infty}^{\infty} |\, u \,| \, \mathrm{d}x \right)^{\frac{1}{q+1}} \left(\int_{-\infty}^{\infty} |\, u \,|^{q-1} \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x \right)^{\frac{1}{q+1}} \quad \big(t \ge 0\big). \tag{3.6}$$

Further substituting (3.6) into

$$\| u(t) \|_{L^q}^q \le \| u(t) \|_{L^\infty}^{q-1} \| u(t) \|_{L^1}, \tag{3.7}$$

then, (3.7) becomes

$$\| u(t) \|_{L^q}^q \le \left(\frac{q+1}{2} \right)^{\frac{2(q-1)}{q+1}} \left(\int_{-\infty}^{\infty} |\, u \,| \, \mathrm{d}x \right)^{\frac{2q}{q+1}} \left(\int_{-\infty}^{\infty} |\, u \,|^{q-1} \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x \right)^{\frac{q-1}{q+1}} \quad \big(t \ge 0\big). \tag{3.8}$$

From (3.8), we immediately get (1).

Thus, the proof of Lemma 3.3 is completed.

By using Lemma 3.3, we show the time-weighted L^q-energy estimates to u with $1 < q < \infty$ stated as the next proposition.

Proposition 3.4 (*Time-weighted L^q-energy Estimates*)**.** *Assume that the same assumptions in Theorem 1.3. For any* $q \in (1, \infty)$*, there exist positive constants* α *and* $C_{\alpha,q}$ *such that the unique global in time solution* u *of the Cauchy problem* (1.1) *has the following uniform energy estimate*

$$\begin{aligned}
&(\,1+t\,)^\alpha \| u(t) \|_{L^q}^q + \int_0^t (\,1+\tau\,)^\alpha \int_{-\infty}^{\infty} |\, u \,|^{q-1} \, \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x\mathrm{d}\tau \\
&+ \int_0^t (\,1+\tau\,)^\alpha \int_{-\infty}^{\infty} |\, u \,|^{q+\gamma-1} \, \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x\mathrm{d}\tau \\
&\le C_{\alpha,q} \| u_0 \|_{L^q}^q + C_{\alpha,q} \int_0^t (\,1+\tau\,)^{\alpha - \frac{q+1}{2}} \left(\int_{-\infty}^{\infty} |\, u \,| \, \mathrm{d}x \right)^q \mathrm{d}\tau.
\end{aligned}$$

Proof of Proposition 3.4. We use (2.7) in Section 2, that is,

$$\begin{aligned}
&\frac{1}{q} (\,1+t\,)^\alpha \| u(t) \|_{L^q}^q + \mu\,(\,q-1\,) \int_0^t (\,1+\tau\,)^\alpha \int_{-\infty}^{\infty} |\, u \,|^{q-2} \, \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x\mathrm{d}\tau \\
&+ \mu\,(\,q-1\,)(\,\gamma+1\,) \int_0^t (\,1+\tau\,)^\alpha \int_{-\infty}^{\infty} |\, u \,|^{q+\gamma-2} \, \big| \, \partial_x u \, \big|^2 \, \mathrm{d}x\mathrm{d}\tau \\
&= \frac{1}{q} \| u_0 \|_{L^q}^q + \frac{\alpha}{q} \int_0^t (\,1+\tau\,)^{\alpha-1} \| u(\tau) \|_{L^q}^q \, \mathrm{d}\tau
\end{aligned} \tag{3.9}$$

for $\alpha > 0$ and $q > 1$. By using Lemma 3.3, the second term on the right-hand side of (3.9) is estimated as

$$
\begin{aligned}
&\frac{\alpha}{q}\int_0^t (1+\tau)^{\alpha-1}\|u(\tau)\|_{L^q}^q\,\mathrm{d}\tau\\
&\le \int_0^t (1+\tau)^{\alpha}\left(\int_{-\infty}^{\infty}|u|^{q-2}|\partial_x u|^2\,\mathrm{d}x\right)^{\frac{q-1}{q+1}}\left(C_{\alpha,q}(1+\tau)^{-\frac{q+1}{2}}\left(\int_{-\infty}^{\infty}|u|\,\mathrm{d}x\right)^{q}\right)^{\frac{2}{q+1}}\mathrm{d}\tau \qquad (3.10)\\
&\le \epsilon\int_0^t (1+\tau)^{\alpha}\int_{-\infty}^{\infty}|u|^{q-2}|\partial_x u|^2\,\mathrm{d}x\mathrm{d}\tau + C_{\alpha,q,\epsilon}\int_0^t (1+\tau)^{\alpha-\frac{q+1}{2}}\left(\int_{-\infty}^{\infty}|u|\,\mathrm{d}x\right)^{q}\mathrm{d}\tau
\end{aligned}
$$

for any $\epsilon > 0$. Choosing ϵ suitably small, substituting (3.10) into (3.9), we obtain the desired time-weighted L^q-energy estimate to u with $1 < q < \infty$.

Thus, the proof of Proposition 3.4 is completed.

Finally, in this section, we prove Theorem 1.3.

Proof of Theorem 1.3. By choosing α suitably large, using Proposition 2.1 in Section 2 to the inequality in Proposition 3.4 and further using the L^1-estimate, that is, Proposition 3.2, we consequently obtain the L^q-estimate for u, that is,

$$
\|u(t)\|_{L^q} \le C_{q,u_0}(1+t)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \quad \bigl(t \ge 0\bigr) \qquad (3.11)
$$

for $q \in [1, \infty)$.

We finally show the L^∞-estimate. By substituting (3.11) into (2.10) in Section 2, we obtain the L^∞-estimate for u, that is,

$$
\|u(t)\|_{L^\infty} \le C_{q,u_0}(\theta)(1+t)^{-\frac{1}{2}+\theta} \quad \bigl(t \ge 0\bigr) \qquad (3.12)
$$

for any $0 < \theta \le 1$.

Thus, the proof of Theorem 1.3 is completed.

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