初等的な２階線形常微分方程式と関連する話題

Elementary Second Order Linear Ordinary Differential Equations and Related Topics

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Abstract. In this paper, we suggest teaching materials of elementary ordinary differential equations and related topics. We particularly treat second order linear ordinary differential equations with variable coefficients.

要旨：本論文では、初等的な常微分方程式に関する教材及び関連する話題を提案する。特に変数係数２階線形常微分方程式について扱う。

1. Introduction.

We consider second order linear ordinary differential equations and related topics for post-secondary education for mathematical analysis, which can be solved only by using some simple technique such as the reduction of order.

We recall that a n-th order linear ordinary differential equation (or linear ordinary differential equation of order n) generally takes the form

$$x^{(n)} + a_1(t)\,x^{(n-1)} + \cdots + a_{n-1}(t)\,x' + a_n(t)\,x = b(t), \tag{1.1}$$

where $x = x(t)$ is an unknown function, x', x'', ... , $x^{(n-1)}$ and $x^{(n)}\big(= \mathrm{d}^n x/\mathrm{d}t^n \big)$ $(n \in \mathbb{N})$ are the ordinary derivatives, and a_1, ... , a_{n-1} , a_n and b are given functions. The term $b = b(t)$ is the so-called inhomogeneous term (or external force term). Therefore, if $b \not\equiv 0$ ($b \equiv 0$), then the equation (1.1) is said to be the inhomogeneous (homogeneous, respectively) linear ordinary differential equation with variable coefficients. Also when the coefficients a_1, ... , a_{n-1} , a_n are constants, (1.1) is also said to be the linear ordinary differential equation with constant coefficients.

For $n = 1$, (1.1) is said to be the inhomogeneous first order linear ordinary differential equation which is rewrited as

$$x' + a(t)\,x = b(t). \tag{1.2}$$

Multiplying (1.2) by an integrating factor $\mathrm{e}^{\int a(t)\,\mathrm{d}t}$ and integrating the resultant formula, we obtain the following general solution $x_g = x_g(t)$ to (1.2) as

$$x_g(t) = \mathrm{e}^{-\int a(t)\,\mathrm{d}t} \int b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}\,\mathrm{d}t + C\,\mathrm{e}^{-\int a(t)\,\mathrm{d}t}, \tag{1.3}$$

in particular,

$$x_g(t) = C\,\mathrm{e}^{-\int a(t)\,\mathrm{d}t}, \tag{1.4}$$

for homogeneous case $b \equiv 0$, where C is an arbitrary constant (complex in general).

For $n = 2$, (1.1) is said to be the inhomogeneous second order linear ordinary differential equation which is also rewrited as

$$x'' + p(t)\,x' + q(t)\,x = b(t), \tag{1.5}$$

where p, q and b are given functions.

We are going to obtain the general solution to (1.5). To do that, we consider the corresponding homogeneous equation

$$x'' + p(t)\,x' + q(t)\,x = 0, \tag{1.6}$$

and use a technique of reduction of order which is the so-called d'Alembert method or d'Alembert reduction (see the following processes (1.7)-(1.10) and (1.14)-(1.20)).

Let $x_1 = x_1(t)$ be one nontrivial solution (particular solution) to (1.6), which depends on q in general. Then, we can put a new unknown function $u = u(t)$ as

$$x = x_1 \cdot u, \tag{1.7}$$

substitute (1.7) into (1.5) and rewrite (1.5) as

$$x_1(t)\,u'' + \big(\,2\,x_1'(t) + p(t)\,x_1(t)\,\big)\,u' = b(t). \tag{1.8}$$

Further putting a new unknown function $v = v(t)$ as

$$v = u' \tag{1.9}$$

and substituting (1.9) into (1.8), then (1.8) becomes the following first order homogeneous linear ordinary differential equation.

$$x_1(t)\,v' + \big(\,2\,x_1'(t) + p(t)\,x_1(t)\,\big)\,v = b(t). \tag{1.10}$$

Then, (1.10) and (1.8) are easily solved by using (1.3) as

$$v(t) = \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t} \int \frac{b(t)}{x_1(t)}\,\mathrm{e}^{\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t + C_1\,\mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}, \tag{1.11}$$

and

$$\begin{aligned} u(t) = &\int \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t} \left(\int \frac{b(t)}{x_1(t)}\,\mathrm{e}^{\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t \right) \mathrm{d}t \\ &+ C_1 \int \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t + C_2, \end{aligned} \tag{1.12}$$

respectively.

Thus, from (1.7) and (1.12), the desired general solution $x_g = x_g(t)$ to (1.5) is obtained as

$$\begin{aligned} x_g(t) = x_1(t) &\int \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t} \left(\int \frac{b(t)}{x_1(t)}\,\mathrm{e}^{\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t \right) \mathrm{d}t \\ &+ C_1\,x_1(t) \int \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t + C_2\,x_1(t), \end{aligned} \tag{1.13}$$

where C_1 and C_2 are arbitrary constants.

On the other hand, we substitute (1.7) into (1.6) (instead of (1.5)) and rewrite (1.6) as

$$x_1(t)\,u'' + \big(\,2\,x_1'(t) + p(t)\,x_1(t)\,\big)\,u' = 0. \tag{1.14}$$

Further putting a new unknown function $v = v(t)$ as

$$v = u' \tag{1.15}$$

and substituting (1.15) into (1.14), then (1.14) becomes the following first order homogeneous linear ordinary differential equation.

$$x_1(t)\,v' + \big(\,2\,x_1'(t) + p(t)\,x_1(t)\,\big)\,v = 0. \tag{1.16}$$

Then, (1.16) and (1.14) are easily solved by using (1.4) as

$$v(t) = C_1\,\mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t} \tag{1.17}$$

and

$$u(t) = C_1 \int \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t + C_2, \tag{1.18}$$

respectively. From (1.7) and (1.18), we obtain the desired general solution $x = x_h(t)$ to (1.6) as follows.

$$x_h(t) = C_1\,x_1(t) \int \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t + C_2\,x_1(t), \tag{1.19}$$

where C_1 and C_2 are arbitrary constants.

Therefore, if $x_p = x_p(t)$ is one particular solution to (1.5), then the general solution $x_g = x_g(t)$ to (1.5) also has the following form.

$$x_g(t) = x_p(t) + x_h(t) = x_p(t) + C_1\,x_1(t) \int \mathrm{e}^{-\int \frac{2\,x_1'(t)+p(t)\,x_1(t)}{x_1(t)}\,\mathrm{d}t}\,\mathrm{d}t + C_2\,x_1(t), \tag{1.20}$$

where C_1 and C_2 are arbitrary constants.

However, if the coefficients $a_1, \ldots, a_{n-1}, a_n$ are constants, we can solve (1.1) (therefore, (1.5) and (1.6)) easier by the method with differential operator (see the following Theorem 1.1 and Theorem 1.2). Using the differental operator $D_t := \mathrm{d}/\mathrm{d}t$ with the n-th degree polynomial function

$$P_n(x) := x^n + a_1\,x^{n-1} + \cdots + a_{n-1}\,x + a_n,$$

we rewrite (1.1) as

$$P_n(D_t)\,x = b(t), \tag{1.21}$$

where

$$P_n(D_t) = D_t^n + a_1\,D_t^{n-1} + \cdots + a_{n-1}\,D_t + a_n, \qquad D_t^k = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k = \frac{\mathrm{d}^k}{\mathrm{d}t^k} \quad \big(k \in \{1,\,2,\,\cdots,\,n\}\big).$$

For simplicity, we now treat the homogeneous case

$$P_n(D_t)\,x = 0. \tag{1.22}$$

If $P_n(D_t) = \prod_{k=1}^{n} (\,D_t - \alpha_k\,)$ or $P_n(D_t) = (\,D_t - \alpha\,)^n$ where $\alpha,\,\alpha_k \in \mathbb{C}$, then the general solution to (1.22) is given in the well-known theorems (for the proofs, see [3] and so on, see also [5]).

Theorem 1.1. *Let* $\alpha_k \neq \alpha_l \ \big(\alpha_k,\, \alpha_l \in \mathbb{C}\big)$ *for any* $k,\, l \in \{1,\, 2,\, \cdots,\, n\}$ $(n \in \mathbb{N})$. *Then, the* n*-th order homogeneous linear ordinary differential equation*

$$\prod_{k=1}^{n} (\, D_t - \alpha_k \,)\, x = 0$$

has the following general solution

$$x(t) = \sum_{k=1}^{n} C_k \, \mathrm{e}^{\alpha_k t},$$

where $C_1,\, C_2,\, \cdots,\, C_n$ *are arbitrary constants.*

Theorem 1.2. *The* n*-th order homogeneous linear ordinary differential equation*

$$(\, D_t - \alpha \,)^n \; x = 0 \quad \big(\alpha \in \mathbb{C},\, n \in \mathbb{N}\big)$$

has the following general solution

$$x(t) = \sum_{k=1}^{n} C_k \, t^{k-1} \, \mathrm{e}^{\alpha t},$$

where $C_1,\, C_2,\, \cdots,\, C_n$ *are arbitrary constants.*

In the following sections, we give some examples of teaching materials for higher order linear ordinary equations by using the reduction of order.

2. Examples for higher order nonlinear ordinary differential equations.

In this section, we give some examples of teaching materials for higher order linear ordinary differential equations by using the reduction of order, the d'Alembert method (cf.[1, 2, 4-6]).

Example 2.1. Find the general solution of

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 4\,\frac{\mathrm{d}x}{\mathrm{d}t} + 3\,x = 0. \tag{2.1}$$

Solution: We can guess $x_p(t) = \mathrm{e}^t \ (\not\equiv 0)$ is one particular solution to (2.1) from

$$x_p''(t) = x_p'(t) = x_p(t) = \mathrm{e}^t.$$

By using x_p, we seek a general solution to (2.1). We now put a new unknown function $u = u(t)$ as

$$x = x_p \cdot u, \tag{2.2}$$

substitute (2.2) into (2.1) and rewrite (2.1) as

$$u'' - 2\,u' = 0. \tag{2.3}$$

Further putting a new unknown function $v = v(t)$ as

$$v = u' \tag{2.4}$$

and substituting (2.4) into (2.3), then (2.3) becomes the following first order homogeneous linear ordinary differential equation.

$$v' - 2\,v = 0. \tag{2.5}$$

The general solution to (2.5) is easily given as

$$v(t) = C_1\,\mathrm{e}^{2t}. \tag{2.6}$$

From (2.4), integrating (2.6), the general solution to (2.3) is also given as

$$u(t) = C_1\,\mathrm{e}^{2t} + C_2. \tag{2.7}$$

From (2.2), we obtain the desired general solution $x_g = x_g(t)$ to (2.1) as follows.

$$x_g(t) = C_1\,\mathrm{e}^{3t} + C_2\,\mathrm{e}^{t}. \tag{2.8}$$

Remark 2.2. By the method with differential operator, we can solve (2.1) easier because (2.1) is a homogeneous linear equation with constant coefficients. In fact, we can rewrite (2.1) as

$$\left(D_t^2 - 4\,D_t + 3 \right) x = \left(D_t - 3 \right)\left(D_t - 1 \right) x = 0. \tag{2.9}$$

Therefore, by using Theorem 1.1 with (2.9), we immediately have (2.8).

Example 2.3. Find the general solution of

$$\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} - 4\,\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + 3\,\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0. \tag{2.10}$$

Solution: Putting $x = x(t)$ as $x = y''$, (2.10) becomes (2.1). Therefore, integrating (2.8) twice, we obtain the desired general solution to (2.10) as follows.

$$y_g(t) = \iint x_g(t)\,(\mathrm{d}t)^2 = C_1\,\mathrm{e}^{3t} + C_2\,\mathrm{e}^{t} + C_3\,t + C_4.$$

Example 2.4. Find the general solution of

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 2\,a\,t\,\frac{\mathrm{d}x}{\mathrm{d}t} - 2\,a\,x = 0, \tag{2.11}$$

where a is a constant.

Solution: We can guess $x_p(t) = \mathrm{e}^{at^2}$ $(\not\equiv 0)$ is one particular solution to (2.11) from

$$x_p'(t) = 2\,a\,t\,\mathrm{e}^{at^2} = 2\,a\,t\,x_p(t), \qquad x_p''(t) = 2\,a\,x_p(t) + 4\,a^2\,t^2\,x_p(t).$$

Putting a new unknown function $u = u(t)$ as

$$x = x_p \cdot u \tag{2.12}$$

and substitute (2.12) into (2.11) gives

$$u'' + 2\,a\,t\,u' = 0. \tag{2.13}$$

Further putting a new unknown function $v = v(t)$ as

$$v = u' \tag{2.14}$$

and substituting (2.14) into (2.13), then (2.13) becomes the following first order homogeneous linear ordinary differential equation.

$$v' + 2\,a\,t\,v = 0. \tag{2.15}$$

The general solution to (2. 15) is easily given as

$$v(t) = C_1\,\mathrm{e}^{-at^2}. \tag{2.16}$$

From (2.14), integrating (2.16), the general solution to (2.13) is also given as

$$u(t) = C_1 \int \mathrm{e}^{-at^2}\,\mathrm{d}t + C_2. \tag{2.17}$$

From (2.12), we obtain the desired general solution $x_g = x_g(t)$ to (2.11) as follows.

$$x_g(t) = C_1\,\mathrm{e}^{at^2} \int \mathrm{e}^{-at^2}\,\mathrm{d}t + C_2\,\mathrm{e}^{at^2}.$$

Example 2.5. Find the general solution of

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + t\,\frac{\mathrm{d}x}{\mathrm{d}t} - x = t^2. \tag{2.18}$$

Solution: At first, we look for one particular solution to the homogeneous equation corresponded to (2.18) as follows.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + t\,\frac{\mathrm{d}x}{\mathrm{d}t} - x = 0. \tag{2.19}$$

Putting $x_1(t\,;\,\alpha) = t^{\alpha}$, where α is a constant which is suitably chosen after, and substituting it into (2.19), we have

$$\left(D_t^2 + t\,D_t - 1 \right) x_1(t\,;\,\alpha) = \alpha\,(\alpha - 1)\,t^{\alpha-2} + (\alpha - 1)\,t^{\alpha} = 0. \tag{2.20}$$

From (2.20), we choose $\alpha = 1$. Then $x_1(t) = x_1(t\,;\,1) = t\,(\not\equiv 0)$ is one particular solution to (2.19). By using x_1, we put a new unknown function $u = u(t)$ as

$$x = x_1 \cdot u, \tag{2.21}$$

substitute (2.21) into (2.19) and rewrite (2.18) as

$$t\,u'' + \left(2 + t^2 \right) u' = t^2. \tag{2.22}$$

Further putting a new unknown function $v = v(t)$ as

$$v = u' \tag{2.23}$$

and substituting (2.23) into (2.22), then (2.22) becomes the following first order inhomogeneous linear ordinary differential equation.

$$t\,v' + \left(2 + t^2\right) v = t^2. \tag{2.24}$$

The general solution to (2.24) is easily given as

$$\begin{aligned} v(t) &= t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}} \int t^3\,\mathrm{e}^{\frac{t^2}{2}}\,\mathrm{d}t + C_1\,t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}} \\ &= t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}} \left(t^2 - 2\right) \mathrm{e}^{\frac{t^2}{2}} + C_1\,t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}} \\ &= 1 - 2\,t^{-2} + C_1\,t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}}. \end{aligned} \tag{2.25}$$

From (2.23), integrating (2.25), the general solution to (2.22) is also given as

$$u(t) = t + 2\,t^{-1} + C_1 \int t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t + C_2. \tag{2.26}$$

Using (2.26) with (2.21), we obtain the desired general solution $x_g = x_g(t)$ to (2.18) as follows.

$$x_g(t) = t^2 + 2 + C_1\,t \int t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t + C_2\,t. \tag{2.27}$$

Remark 2.6. By using one particular solution $x_p(t) = t^2 + 2\,(\not\equiv 0)$ to (2.18) and the general solution to (2.19) as

$$x_h(t) = C_1\,t \int t^{-2}\,\mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t + C_2\,t \tag{2.28}$$

from (1.19) in Section 1 with $x_1(t) = t$, $p(t) = t$ and $q(t) = -1$, we can also obtain (2.27) from

$$x_g(t) = x_p(t) + x_h(t).$$

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