Original Report

An Analysis of E-B Hysteresis in Positive Column

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Abstract

A long positive column lies in strong magnetic fields $\boldsymbol{B}$ parallel to its axis. The discharge field $\boldsymbol{E}$ shows some difference in magnitude between as increasing and then decreasing $\boldsymbol{B}$. This E-B hysteresis is due to a degree of plasma turbulence in the column. The effect of turbulence is analyzed by the nonlinearity of helical waves and by the rise of ion temperature. The present calculation gives a curve of the E-B hysteresis in relatively good agreement with experimental results. The experiments are made by using the helium gas, of which pressure is less than *0.3* mmHg.

§1. Introduction

In the 1920's Tonks and Langmuir *et al*[1)~3)] investigated the discharge of cylindrical positive column without an applied magnetic field. The behaviour of the positive column was well understood by them both theoretically and experimentally. Later, Bikerton and von Engel[4)] applied the magnetic field $\boldsymbol{B}$ to the positive column. The field $\boldsymbol{B}$ was homogeneous and parallel to the axis of column. The discharge tube was so long that its end effect could be neglected[5)]. They observed that the discharge electric field $\boldsymbol{E}$ decreases as the field $\boldsymbol{B}$ increases. This character is explained by the classical theory of diffusion. That is, the field $\boldsymbol{B}$ tends to make the charged particles frozen in. Then, the diffusion-loss of those particles decreases and simultaneously the energy supply to maintain the ion and electron densities is ready in a less degree. This brings the decrease of $\boldsymbol{E}$.

The above experiment was farther investigated by Lehnert and Hoh[6),7)] applying considerable strong fields $\boldsymbol{B}$. They found the increase of $\boldsymbol{E}$ when $\boldsymbol{B}$ exceeded a certain critical value B_c This increase of $\boldsymbol{E}$ is caused by the increase of diffusion which comes from a plasma instability in the positive column. This instability is known as due to helical oscillations.

Bohm *et al*[8)] pointed out that the density fluctuations of charged particles produce the field fluctuations, which give rise to drift motions of charged particles across the field $\boldsymbol{B}$. Those random drift motions lead to a new type of the diffusion, which is called "drain diffusion" or "anomalous diffusion".

Kadomtsev and Nedospasov[9)] quantitatively explained the characteristics of E-B curve for $B>B_c$ by the helical instability. They superimposed helical perturbations upon the plasma density in stable state. The existence of helical disturbance was confirmed by the experiments of Paulikas and Pyle[10)] at the vicinity of B_c. The occurrence of helical instability in $B \gtrsim B_c$ is explained as follows: The Plasma in the column is rotating for the action of both the field $\boldsymbol{B}$ in the axis direction and the ambipoler field $\boldsymbol{E}_a$ in the radius direction. Any disturbance of the plasma density diffuses out by many collisions of neutral particles in the region $B<B_c$, i. e. it is stable. However, in $B \gtrsim B_c$ the density disturbance is maintained owing to the tendency frozen in the strong field $\boldsymbol{B}$. This disturbance flows along the discharge field $\boldsymbol{E}$ with the motion of the plasma rotation explained above.

Lehnert and Hoh[7)] observed that the critical field

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$\boldsymbol{B}_c$ slightly depends on the discharge current $\boldsymbol{I}$. Matsumoto[11] theoretically analyzed this effect by using a parameter ϕ which is inversely proportional to the current $\boldsymbol{I}$. His result was in good agreement with the experiments of Lehnert and Hoh[7]. Sato *et al*[12] reported the hysteresis of E-B characteristics in the helium positive column. The hysteresis appears when the helium gas pressure is less than 0.3 mmHg.

At the present paper we analyze the E-B hysteresis mentioned above. Following Kadomstev *et al*, we consider the finite amplitudes of helical waves and calculate their nonlinear effect on the diffusion of charged particles. Especially, the effect of ion temperature is introduced into our analysis.

§2. Basic Equations

We consider the helium positive column which is enough long to neglect the end effect. The external magnetic field $\boldsymbol{B}$ exists parallel to the axis of column. Accoding to Lehnert[6] and Matsumoto[11),13)], eqs. (1)～(4) are valid under the following conditions :

(*a*) The mean free paths of charged particles are small compared to the tube radius R.

(*b*) The production rate ξn_e of charged particles is proportional to the electron density n_e, where ξ is a function of the electron temperature and the neutral gas density. The value ξ indicates the number of ionization events per electron per unit time.

(*c*) The discharge current is so weak that the magnetic field induced by its current is negligible.

(*d*) The ionization degree is very low and at highest 1%.

(*e*) The electron attachment can be neglected.

(*f*) The frequency of helical wave is much lower than the mean collision frequencies ν_j between neutral and charged particles, where j indicates the j-type particle, *i. e.* $j=i$ for ion and $j=e$ for electron.

(*g*) The macroscopic velocity $\boldsymbol{V}_j$ is small compared to the thermal velocity of j-type particle.

The condition (*c*) guarantees that the mutual interaction of charged particles is negligible and the frictional coupling between neutral and charged particles does not induce an appreciable motion of the neutral gas. The helium gas fulfiles the condition (*e*). From the conditions (*a*) and (*g*), we can assume that the pressure P_j is scalar and the temperature T_j uniform.

Averaging the Boltzmann equation over the velocity space, we obtain

$$\frac{\partial n_i}{\partial t}+\nabla\cdot\boldsymbol{\Gamma}_i=\xi n_e \tag{1}$$

$$\frac{\partial n_e}{\partial t}+\nabla\cdot\boldsymbol{\Gamma}_e=\xi n_e \tag{2}$$

Here, n_j and $\boldsymbol{\Gamma}_j$ represent the number density and flow density of the j-type particles. They are defined as

$$n_j(\boldsymbol{r},\ t)=\int d^3\boldsymbol{v} f_j(\boldsymbol{r},\ \boldsymbol{v},\ t)$$

$$\boldsymbol{\Gamma}_j(\boldsymbol{r},\ t)=\int d^3\boldsymbol{v}\boldsymbol{v} f_j(\boldsymbol{r},\ \boldsymbol{v},\ t)$$

where $f_j(\boldsymbol{r},\ \boldsymbol{v},\ t)$ is the distribution function of the j-type particles. Multiplying the Boltzmann equation by $\boldsymbol{v}$ and averaging it over the $\boldsymbol{v}$-space, we obtain the followng kinetic equations.

$$en_i\boldsymbol{E}+e\boldsymbol{\Gamma}_i\times\boldsymbol{B}-\nabla P_i-\nu_i m_i\boldsymbol{\Gamma}_i=0, \tag{3}$$

$$en_e\boldsymbol{E}+e\boldsymbol{\Gamma}_e\times\boldsymbol{B}+\nabla P_e+\nu_e m_e\boldsymbol{\Gamma}_e=0, \tag{4}$$

where $e=1.6\times10^{-19}$ coulomb and m_j is the mass of the j-type particle.

We assume the isothermal change for the charged particle density, *i. e.*

$$\nabla P_j=\kappa T_j\nabla n_j \tag{5}$$

Here, $\kappa=1.38\times10^{-23}JK^{-1}$ is the Boltzmann constant. We solve eqs. (3) and (4) for $\boldsymbol{\Gamma}_j$ in the cylindrical coordinates $(r,\ \theta,\ z)$, and substitute those solutions into the continuity equations (1) and (2).

$$\frac{\partial n_i}{\partial t}-\xi n_e+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{n_i\mu_{i\perp}(E_r+s_iE_\theta)-D_{i\perp}\frac{\partial n_i}{\partial r}\right\}\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[n_i\mu_{i\perp}(-s_iE_r+E_\theta)-\frac{D_{i\perp}}{r}\frac{\partial n_i}{\partial\theta}\right]+\frac{\partial}{\partial z}\left\{n_i\mu_iE_z-D_i\frac{\partial n_i}{\partial z}\right\}=0, \tag{6}$$

$$\frac{\partial n_e}{\partial t}-\xi n_e+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{-n_e\mu_{e\perp}(E_r-s_eE_\theta)-D_{e\perp}\frac{\partial n_e}{\partial r}\right\}\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[-n_e\mu_{e\perp}(s_eE_r+E_\theta)-\frac{D_{e\perp}}{r}\frac{\partial n_e}{\partial\theta}\right]+\frac{\partial}{\partial z}\left\{n_e\mu_eE_z+D_e\frac{\partial n_e}{\partial z}\right\}=0. \tag{7}$$

Equation (5) was used in the above derivation.

Here, $\mu_j=e/m_j\nu_j$ the mobility, $D_j=\kappa T_j\mu_j/e$ the diffusion constant, $s_j=\Omega_j/\nu_j$, $\Omega_j=eB/m_j$ the Larmor frequency, $\mu_{j\perp}=\mu_j/(1+s_j^2)$ and $D_{j\perp}=D_j/(1+s_j^2)$.

§ 3. Perturbation and Nonlinear Effect

We write n_{j1} and E_1 as the perturbations from the coresponding average quantities n_{j0} and E_0. The density perturbations n_{j1} are assumed to be of the following helical form.

$$n_{i1}=Re\{HJ_1(\lambda_1 r/R)e^{i(kz+m\theta-\omega t)}\}, \tag{8}$$

$$n_{e1}=Re\{GJ_1(\lambda_1 r/R)e^{i(kz+m\theta-\omega t)}\}, \tag{9}$$

where $Re\{ //\}$ indicates the real part, H and G the complex numbers and $\lambda_1=3.8317$ the first root of the Bessel function $J_1(x)=0$, In the present analysis we confine ourselves to $|m|=1$. The perturbed potential V_1 which gives $E_1=-\nabla V_1$ is subject to the Poisson equation.

$$\nabla^2 V_1=-\frac{e}{\varepsilon_0}(n_{i1}-n_{e1}),\quad \text{for } r\leq R, \tag{10}$$

$$=0 \qquad ,\quad \text{for } r>R, \tag{11}$$

where $\varepsilon_0=8.854\times10^{-12}\mathrm{Fm}^{-1}$ is the dielectric constant. We solve the above equations under the boundary conditions[11],

$$V_1(r=0)=0 \text{ and } V_1(r=R)=0. \tag{12}$$

The solution is given as follows ;

$$V_1=Re\left\{\frac{e}{\varepsilon_0}\frac{H-G}{\beta_1^2+k^2}J_1(\beta_1 r)e^{i(kz+m\theta-\omega t)}\right\} \tag{13}$$

Here, $\beta_1=\lambda_1/R$. The perturbed electric field E_1 is derived from the above solution (13).

$$E_{r1}=-\frac{\partial V_1}{\partial r}=Re\left\{-\frac{e}{\varepsilon_0}\frac{H-G}{\beta_1^2+k^2}\frac{1}{R}\frac{\partial J_1(\lambda_1 x)}{\partial x}e^{i(kz+m\theta-\omega t)}\right\}, \tag{14}$$

$$E_{\theta1}=-\frac{1}{r}\frac{\partial V_1}{\partial\theta}=Re\left\{-\frac{im}{r}\frac{e}{\varepsilon_0}\frac{H-G}{\beta_1^2+k^2}J_1(\lambda_1 x)e^{i(kz+m\theta-\omega t)}\right\}, \tag{15}$$

$$E_{z1}=-\frac{\partial V_1}{\partial z}=Re\left\{-ik\frac{e}{\varepsilon_0}\frac{H-G}{\beta_1^2+k^2}J_1(\lambda_1 x)e^{i(kz+m\theta-\omega t)}\right\}, \tag{16}$$

where $x=r/R$. We substitute eqs. (14)～(16) into eqs. (6) and (7) in which $E=E_0+E_1$ are expressed in the cylindrical coordinates $(r,\ \theta,\ z)$. We take account of the nonlinear terms, which come from $n_{j1}E_1$, in the time average. Those are defined as

$$a_{i1}=\langle n_{i1}E_1\rangle, \tag{17}$$

$$a_{e1}=\langle n_{e1}E_1\rangle. \tag{18}$$

Using eqs. (8), (9), (14)～(16), we can write the above a_{j1} as

$$a_{ir}=\frac{e}{2\varepsilon_0}\frac{Re\{H^*(G-H)\}}{\beta_1^2+k^2}\frac{J_1(\lambda_1 x)}{R}\frac{\partial J_1(\lambda_1 x)}{\partial x}, \tag{19}$$

$$a_{er}=\frac{e}{2\varepsilon_0}\frac{Re\{G^*(G-H)}{\beta_1^2+k^2}\frac{J_1(\lambda_1 x)}{R}\frac{\partial J_1(\lambda_1 x)}{\partial x}, \tag{20}$$

$$a_{i\theta}=a_{e\theta}=-\frac{me}{2\varepsilon_0}\frac{Re\{iG^*H\}}{\beta_1^2+k^2}\frac{J_1^2(\lambda_1 x)}{x}, \tag{21}$$

$$a_{iz}=a_{ez}=-\frac{ke}{\varepsilon_0}\frac{Re\{iG^*H\}}{\beta_1^2+k^2}J_1^2(\lambda_1 x). \tag{22}$$

Here, H^* and G^* represent the complex conjugates. The results $a_{i_\theta}=a_{e_\theta}$ and $a_{i_z}=a_{e_z}$ are due to the assumption that only the two modes $m=1,\ -1$ exist,

§ 3. Average Density Distribution and Discharge electric field

The average quantites n_{j_0} and a_j are independent of θ and z for the assumption of axial symmetry and uniformity in the z-direction. Taking the time average of eqs. (6), (7), we obtain

$$-\xi n_{e0}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\mu_{i\perp}(n_{i0}E_{r0}+a_{ir}+s_i a_{i\theta})-D_{i\perp}\frac{\partial n_{i0}}{\partial r}\right\}\right]=0, \tag{23}$$

$$-\xi n_{e0}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{-\mu_{e\perp}(n_{e0}E_{r0}+a_{er}-s_e a_{e\theta})-D_{e\perp}\frac{\partial n_{e0}}{\partial r}\right\}\right]=0. \tag{24}$$

Here, $E_{\theta_0}=0$ was taken from the cylindrial system. We assume $n_{i_0}=n_{e_0}=n_0$ on the neutrality of plasma. The field E_{r_0} in the r-direction is derived from eqs. (23) and (24) by eliminating ξ.

$$E_{r0}=-\frac{D_{e\perp}-D_{i\perp}}{\mu_{e\perp}+\mu_{i\perp}}\frac{1}{n_0}\frac{\partial n_0}{\partial r}+\frac{s_e\mu_{e\perp}-s_i\mu_{i\perp}}{\mu_{e\perp}+\mu_{i\perp}}\frac{a_\theta}{n_0}-\frac{\mu_{e\perp}a_{er}+\mu_{i\perp}a_{ir}}{\mu_{e\perp}+\mu_{i\perp}}\frac{1}{n_0}. \tag{25}$$

This field is called the *ambipolar field* E_a (See § 1).

We are interested in the helical instability at $B\gtrsim B_c$. From the experimental results, we estimate s_e about 20, *i.e.* $s_e^2\gg1$ $(s_e=\Omega_e/\nu_e)$. In our case, s_i^2 are always much less than one. From eqs. (23) and (24), therefore, we obtain

$$\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial n_0}{\partial x}\right)+\lambda^2 n_0=\frac{e}{\kappa T_e}\frac{s_i+s_e}{1+\varepsilon}\frac{R}{x}\frac{\partial}{\partial x}(xa_\theta)$$

$$+\frac{e}{\kappa T_e}\frac{1}{1+\varepsilon}\frac{R}{x}\frac{\partial}{\partial x}\{x(a_{ir}-a_{er})\}. \quad (26)$$

Here, $x=r/R$, $\varepsilon=T_i/T_e$ and

$$\lambda^2=\frac{e\xi R^2(1+y)}{\kappa T_e\mu_i(1+\varepsilon)},\ y=s_is_e. \quad (27)$$

To simplify eq. (26), we put $n_0(r)=Nh(x)$ and

$$\frac{G-H}{H}=\sigma+im\eta \quad (28)$$

in the definition of a_j (See eqs. (17)~(22)). Here, N is the number density of charged particles at the z-axis *i.e.* $N\equiv n(0)$. We consider that σ, $\eta\ll 1$ and $|kR|\ll\lambda_1$ for the helical instability. By the above considerations and by using eq. (19)~(22), eq. (26) leads to

$$\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial h}{\partial x}\right)+\lambda^2h=\frac{A_1}{x}\frac{\partial}{\partial x}\mathrm{J}_1{}^2(\lambda_1x), \quad (29)$$

where

$$A_1=-\frac{s_e}{\kappa T_e}\frac{1}{1+\varepsilon}\frac{e^2}{2\varepsilon_0}\frac{R^2|H|^2}{\lambda_1{}^2}\eta. \quad (30)$$

The solutions $h(x)$ of eq. (29), the form of density distribution, is given as the eigen function of λ (See Appendix).

$$h(x)=\mathrm{J}_0(\lambda x)+\lambda\frac{\pi}{2}\frac{A_1}{N}\times\{-\mathrm{J}_0(\lambda x)B_1(x)+\mathrm{N}_0(x)B_2(x)\}, \quad (31)$$

where

$$B_1(x)=\int_0^x dx\mathrm{J}_1{}^2(\lambda_1x)\mathrm{N}_1(\lambda x), \quad (32)$$

$$B_2(x)=\int_0^x dx\mathrm{J}_1{}^2(\lambda_1x)\mathrm{J}_1(\lambda x). \quad (33)$$

Here, $\mathrm{N}_0(x)$ and $\mathrm{N}_1(x)$ are the Bessel functions of the second kind. From the boundary conditions of density distribution, *i.e.* $h(0)=1$ and $h(1)=0$, the value of λ is dertermined. The conditions $h(0)=1$ is automatically satisfied in eq. (31). Then,

$$h(1)=0=\mathrm{J}_0(\lambda)+\lambda\frac{\pi}{2}\frac{A_1}{N}\times\{-\mathrm{J}_0(\lambda)B_1(1)+\mathrm{N}_0(\lambda)B_2(1)\}. \quad (34)$$

It should be noted that the λ becomes $\lambda_0=2.4048$ for the unperturbed state, λ_0 being the first zero point $\mathrm{J}_0(x)$.

The number density N at the axis is calculated by the discharge current I which is given as

$$I=\int_0^R dr2\pi re\langle n_e\mu_eE_{z0}\rangle=2\pi R^2e\mu_e\left\{\int_0^1 dxxn_0E_{z0}+\int_0^1 dxxa_z\right\}. \quad (35)$$

The second term in the wavy bracket is negligible compared to the first one, and then eq. (35)leads to

$$N=j\frac{273\lambda}{2e\mu_{i0}T_n\mathrm{J}_1(\lambda)}\frac{P}{E_{z0}}-\frac{\pi}{2}\lambda\frac{A_1}{\mathrm{J}_1(\lambda)}\times\{-\mathrm{J}_1(\lambda)B_1(1)+\mathrm{N}_0(\lambda)B_2(1)\}. \quad (36)$$

Here, $j=I/(\pi R^2)$ is the current density and

$$\mu_{i0}=\mu_iP\frac{T_n}{273},$$

T_n being the temperature of the neutral gas. It is convenient to use PR, ω/P, E_{z0}/P, B/P and j, respectively, insted of R, ω, E_{0z}, B and I. If PR and j have same values for two different columns, E_{z_0}/P and ω/P for each column have also same values for a given velue of B/P. It is called the *scale relation.*

The average flow of ion $\langle\Gamma_{ir}\rangle$ is given as the solution of the time averaged equations(3) and (4).

$$\langle\Gamma_{ir}\rangle=-\frac{\mu_{e\perp}D_{i\perp}+\mu_{i\perp}D_{e\perp}}{\mu_{e\perp}+\mu_{i\perp}}\frac{\partial n_0}{\partial r}+\frac{\mu_{e\perp}\mu_{i\perp}(s_i+s_e)}{\mu_{e\perp}+\mu_{i\perp}}a_\theta. \quad (37)$$

Here, we used eq. (25). The coresponding flow of electrons has the same value with $\langle\Gamma_{ir}\rangle$, because of the symmetry of suffies i and e in the above expression. The second term shows the nonlinear effect, which leads to an anomalously large diffusion loss of charged particles across the magnetic field $\boldsymbol{B}$. This increase of diffusion loss is caused by random drifts $(\boldsymbol{E}_{\theta 1}\times\boldsymbol{B})$ of charged particles in the r-direction.

Extending the classical theory of diffusion, we will obtain the magnitude of E_{zo} (discharge electric field) and T_e (electron temperature) as a function of λ and $y=s_is_e$, where $s_j=\Omega_j/\nu_j$ is the ratio of Larmor frequency to collision one. The functional forms of E_{z_0}/P and T_e are experimentally given like that

$$\frac{E_{z0}}{P}=f_A(PR),\ T_e=f_B\left(\frac{E_{z0}}{P}\right) \quad (38)$$

Using Lehnert's form[6),14)] for ξ/P, we rewrite eq. (27) as

$$\frac{\xi}{P}=\frac{\lambda^2}{(PR)^2}\frac{\mu_{i0}\kappa T_e}{e}\frac{T_n}{273}\frac{1+\varepsilon}{1+y} \quad (39)$$

Then, we can derive the following form.

$$\frac{E_{z0}}{P}=f_A\left(PR\frac{\lambda}{\lambda_0}\sqrt{\frac{1+y}{1+\varepsilon}}\right) \quad (40)$$

This equation is used to obtain the discharge electric field E_{z_0}.

§ 4. Instability Region

To determine a region in which the helical instability occurs, we discuss the growth of density perturbations n_{i_1} and n_{e_1} given in eqs. (8) and (9). Our basic equations (6) and (7) lead to the folllowing simultaneous equations for n_{i_1} and n_{e_1}.

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\mu_{i\perp}(n_{i_0}E_{r_1}+n_{i_1}E_{r_0}+s_i n_{i_0}E_{\theta_1}) - D_{i\perp}\frac{\partial n_{i_1}}{\partial r}\right\}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\mu_{i\perp}(-s_i n_{i_0}E_{r_1}-s_i n_{i_1}E_{r_0}+n_{i_0}E_{\theta_1}) - \frac{D_{i\perp}}{r}\frac{\partial n_{i_1}}{\partial r}\right\} + \frac{\partial}{\partial z}\left\{\mu_i n_{i_0}E_{z_1}+\mu_i n_{i_1}E_{z_0}-D_i\frac{\partial n_{i_1}}{\partial z}\right\} + \frac{\partial n_{i_1}}{\partial t}-\xi n_{e_1}=0 \tag{41}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\{-\mu_{e\perp}(n_{e_0}E_{r_1}+n_{e_1}E_{r_0}-s_e n_{e_0}E_{\theta_1}) - D_{e\perp}\frac{\partial n_{e_1}}{\partial r}\}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left\{-\mu_{e\perp}(s_e n_{e_0}E_{r_1}+s_e n_{e_1}E_{r_1}+n_{e_0}E_{\theta_1}) - \frac{D_{e\perp}}{r}\frac{\partial n_{e_1}}{\partial \theta}\right\} - \frac{\partial}{\partial z}\left\{\mu_e n_{e_0}E_{z_1}+\mu_e n_{e_1}E_{z_0}+D_e\frac{\partial n_{e_1}}{\partial z}\right\} + \frac{\partial n_{e_1}}{\partial t}-\xi n_{e_1}=0 \tag{42}$$

Here, eqs. (23) and (24) for the steady state quantities were used. For covencience, We introduce the following dimention-less fields.

$$\hat{E}_{r0}\equiv(1+y)\frac{eR}{\kappa T_e}E_{r0} = -(1+\varepsilon y)\frac{1}{h(x)}\frac{\partial h(x)}{\partial x}+A_3\frac{J_1^2(\lambda_1 x)}{xh(x)} + A_4\frac{J_1(\lambda_1 x)}{h(x)}\frac{\partial J_1(\lambda_1 x)}{\partial x}, \tag{43}$$

$$\hat{E}_{z0}\equiv\frac{eR}{\kappa T_e}E_{z0},$$

where

$$A_3=-\frac{e^2R^2}{2\varepsilon_0\lambda_1^2}\frac{s_e\eta}{N\kappa T_e}|H|^2,$$

$$A_4=-\frac{e^2R^2}{2\varepsilon_0\lambda_1^2}\frac{(1+y)\sigma}{N\kappa T_e}|H|^2.$$

Farther, we substitute the helical perturbations (8) and (9) into eqs. (41) and (42). Making the operation $\int_0^1 dxxJ_1(\lambda_1 x)$ on those equations, we obtain;

$$\{C_1+imC_3s_i\}(G-H) + \left\{\varepsilon\lambda_1^2C_1-\frac{C_4+(1+\varepsilon)\lambda^2C_6}{1+y} + i\left(\frac{mC_6}{s_e}X\hat{E}_{z0}-\frac{C_6\omega}{W\phi y}-\frac{mC_5s_i}{1+y}\right)\right\}G\phi=0 \tag{44}$$

$$\{-C_1+C_2X^2+imC_3s_e\}(G-H) + \left\{(\lambda_1^2+X^2)C_6-\frac{y\lambda^2(1+\varepsilon)C_6+C_4}{1+y} - i\left(\frac{C_6\omega}{W\phi}+mC_6s_eX\hat{E}_{z0}+\frac{mC_5s_e}{1+y}\right)\right\}G\phi=0 \tag{45}$$

Here, $X=ms_ekR$ is taken as the independent variable, $\phi=\varepsilon_0\kappa T_e/e^2NR^2$, $W=e\mu_iN/\varepsilon_0 y$, and

$$C_1=\int_0^1 dx\frac{xJ_1(\lambda_1 x)}{\lambda_1^2}\times\left\{\frac{\partial J_1(\lambda_1 x)}{\partial x}\frac{\partial h(x)}{\partial x}-\lambda_1^2h(x)J_1(\lambda_1 x)\right\}$$

$$C_2=\int_0^1 dx\frac{xh(x)J_1^2(\lambda_1 x)}{\lambda_1^2}$$

$$C_3=\int_0^1 dx\frac{J_1^2(\lambda_1 x)}{\lambda_1^2}\frac{\partial J_1(\lambda_1 x)}{\partial x}$$

$$C_4=\int_0^1 dxxJ_1(\lambda_1 x)\frac{\partial J_1(\lambda_1 x)}{\partial x}\hat{E}_{r0}$$

$$C_5=\int_0^1 dxJ_1^2(\lambda_1 x)\hat{E}_{r0}$$

$$C_6=\int_0^1 dx\frac{J_1^2(\lambda_1 x)}{x}=\frac{J_0^2(\lambda_1)}{2}$$

The above simultaneous equations (44) and (45) must be always satisfied by the arbitrary variables $(G\text{-}H)$ and ϕG which come from the amplitudes of helical perturbations (See eqs. (8) and (9)). Then, putting the determinant of coefficient matrix in those equations equal to zero, we obtain the dispersion relation of helical waves. We are now interested in the instability, which is given by the condition $\mathscr{I}m(\omega)>0$ in the dispersion relation $\omega=\omega(k)$. This condition is explicity written by

$$\zeta(X)\equiv Q_0+Q_1X+Q_2X^2+Q_3X^3+Q_4X^4>0, \tag{46}$$

for any value of X. The coefficients $Q_0\sim Q_4$ are defined as

$$Q_0=-\alpha_1(\alpha_2^2+\alpha_3^2)+\alpha_3\alpha_4+\alpha_2(\alpha_5-y\alpha_6) +y(\alpha_1\alpha_3-\alpha_4),$$

$$Q_1 = -\alpha_2\left(s_e + \frac{y}{s_e}\right)C_6,$$

$$Q_2 = -2\alpha_1(\alpha_3\alpha_7 + \alpha_2\alpha_8) + \alpha_4\alpha_7 + \alpha_3 C_6 + \alpha_8(\alpha_5 - y\alpha_6) + y(\alpha_1\alpha_7 - C_6),$$

$$Q_3 = -\alpha_8\left(s_e + \frac{y}{s_e}\right)C_6\hat{E}_{z0},$$

$$Q_4 = \alpha_7 C_6 - \frac{C_6}{s_e}(\alpha_7^2 + \alpha_8^2).$$

Here,

$$\alpha_1 = \varepsilon\lambda_1^2 C_6 - \frac{C_4 + (1+\varepsilon)\lambda^2 C_6}{1+y}$$

$$\alpha_2 = \frac{s_e C_1 C_2}{C_1^2 + s_i^2 C_3^2}$$

$$\alpha_3 = \frac{y C_3^2 - C_1^2}{C_1^2 + s_i^2 C_3^2}$$

$$\alpha_4 = \frac{C_4 - \lambda^2(1+\varepsilon)y C_6}{1+y} + \lambda_1^2 C_6$$

$$\alpha_5 = -\frac{s_e C_5}{1+y}$$

$$\alpha_6 = -\frac{s_i C_5}{1+y}$$

$$\alpha_7 = \frac{C_1 C_2}{C_1^2 + s_i^2 C_3^2}$$

$$\alpha_8 = -\frac{s_i C_2 C_3}{C_1^2 + s_i^2 C_3^2}$$

The coefficicients Q_j ($j=0, 1, \cdots, 4$) depend on the fields E_{z0} and $\boldsymbol{B}$, and of the parameters G, H and ε, *i. e.*

$$Q_j = Q_j(E_{z0}, \boldsymbol{B}; G, H, \varepsilon).$$

The field $\boldsymbol{B}$ is contained in y, s_i and s_e, and the amplitudes of helical perturbation G and H in the integrals C_1, C_2, C_4 and C_5. For some fixed values of $\boldsymbol{B}$ and of the parmeters, many curves of $\zeta(X)$ are written by changing the field E_{z0}. (See **Fig. 1**). The field E_{z0} which gives the curve touched with the X-axis determines the border line between stable and unstable regions.

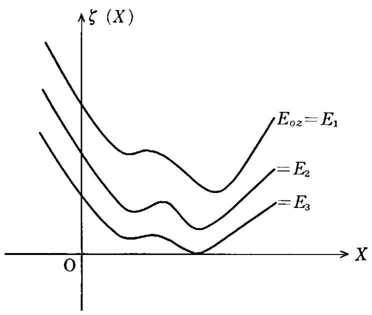

Fig. 1 Curves of $\zeta(X)$ for given values of E_{z0}

§ 5. Result and Discussion

An experimental curve[12] for the E-B hysteresis is shown in **Fig. 2**. The positive column is of the tube radius $R=0.015\ m$ and length $L=2.5\ m$. The experiment is done under the neutral gas pressure $P=0.3\ mmHg$ and the discharge current $I=0.1\ amp$. The abscissa in Fig. 2 is proportional to the field $\boldsymbol{B}$ through s_e. The helical instability sets on at $s_e=s_1$ as increasing $\boldsymbol{B}$. When $\boldsymbol{B}$ decreases from a strong region, the instability ceases at $s_e=s_2$, where $s_2<s_1$. The value of s_1 corresponds to $B=0.0948\ Wb/m^2$ and s_2 to $B=0.0903\ Wb/m^2$.

The theoretical E-B curves in Fig. 2 are drawn for values of $\varepsilon=T_i/T_e$ changed from 0.0 to 0.12 by step 0.04. In the case of increasing $\boldsymbol{B}$, the field E_{z0} follows the E-B curvs for $\varepsilon=0.0$. When the instability sets on, then the perturbed electric fields gives an energy to ions and electrons. However, the energy of electrons is lost by ionization and the electron temperature T_e is maintained nearly constant $(5\sim6)\times10^4$K. On the other hand, the ion temperature T_i increases. This fact is experimentally confirmed[12]. When the amplitudes of perturbed fields are enlarged, the ratio ε increases. Therefore, the field E_{z0} follows back another curve for a higher value of ε, when $\boldsymbol{B}$ decreases from the unstable region. That is, the

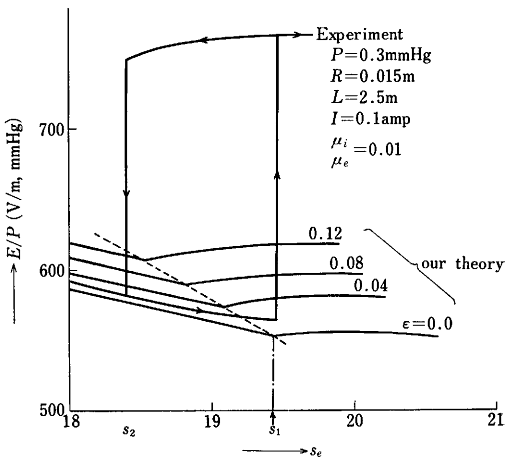

Fig. 2 Theoretical E-B curves for given values of $\varepsilon=T_i/T_e$ and experimental curve of the E-B hysteresis.

hysteresis of *E-B* curve appears.

In the present calculations, the number density N of charged particles at the tube axis is equal to $(4\sim5)\times10^{16}\,\mathrm{m}^{-3}$ and the electron temperature T_e to $(5\sim6)\times10^{4}$K. The frequency of helical wave is about 30 kHz. The value of $|kR|$ is about 0.02, then

$$(h)\quad |kR|\ll\lambda_1=3.8317.$$

We have $s_i s_e(\lambda_D/R)^2\simeq4\times10^{-4}$ and $\omega(R^2/D_{e\perp})(\lambda_D/R)^2\simeq6\times10^{-4}$ in the present analysis. Then, the plasma neutrality conditions[11) :

$$(i)\quad s_i s_e\left(\frac{\lambda_D}{R}\right)^2,\ \omega\left(\frac{R^2}{D_{e\perp}}\right)\left(\frac{\lambda_D}{R}\right)^2\ll1$$

are completely satisfied.

APPENDIX

We obtain the solution of the equation :

$$L(y)\equiv\frac{d}{dx}\left\{P(x)\frac{dy}{dx}\right\}+q(x)y=r(x)$$

Let $y_1(x)$ and $y_2(x)$ be the indepentent solutions of $L(y)=0$. The Green function is defined as

$$G(\xi,\ x)=\begin{cases}\dfrac{y_1(x)y_2(\xi)}{P(\xi)\{y_1(\xi)y_2'(\xi)-y_1'(\xi)y_2(\xi)\}} & (x\leqq\xi)\\[2ex] \dfrac{-y_1(\xi)y_2(x)}{P(\xi)\{y_1(\xi)y_2'(x)-y_1'(\xi)y_2(\xi)\}} & (x\geqq\xi)\end{cases}$$

Then, the solution $y(x)$ is given by

$$y=\int_b^a G(\xi,\ x)r(\xi)d\xi$$
$$=y_1(x)\left[C_1+\int_0^x\frac{y_2(\xi)r(\xi)d\xi}{P(\xi)\{y_1(\xi)y_2'(\xi)-y_1'(\xi)y_2(\xi)\}}\right]$$
$$+y_2(x)\left[C_2+\int_0^x\frac{y_1(\xi)r(\xi)d\xi}{p(\xi)\{y_1(\xi)y_2'(\xi)-y_1'(\xi)y_2(\xi)\}}\right].$$

Here, C_1 and C_2 are the integral constants which are determined by the boundary condition.

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