Original Report

Multi-Threshold Element Networks

Atsumi IMAMIYA*, Shoichi NOGUCHI** and Juro OIZUMI***

(Received August 27, 1976)

Abstact

Any switching function can be realized by a single k-multithreshold element (k-MTE) having a sufficiently large k. However, if for a certain switching function the number of thresholds becomes very large k, then the practical realization of such k-MTE often presents serious difficulties. One alternative to this is to realize the switching function by a p-MTE network in which each MTE has p thresholds ($p<k$).

In this paper, a study of p-MTE networks with different modes of interconnection has been made. Synthesis procedures are developed for a cascade, a 2-level and a feedforward MTE networks in which each MTE has the identical weight vector for the independent (input) variables.

The synthesis precedure for a switching function by cascade p-MTE network in which each MTE may have the different weight vector for the inputs has also been given. This procedure converges for realizing any switching function. Finally, the upper bound for the maximum number of p—MTE needed to realize an arbitrary n-argument switching function is shown.

keywcrds

logical design,
multi-threshold element (MTE),
switching function,
p-MTE network,
multi-threshold function,
CW MET network,
effective threshold,
weight-threshold vector.

I. Introduction

Because they are potentially powerful logical elements, multithreshold elements (MTE's) have recently received a considerable amount of study in logical design [1-9].

This paper is conecrned with a generalization of the conventional networks of threshold elements (TE's) in which p-thresholds ($p=1, 2, \cdots$), rather than the usual single threshold or 3-thresholds [5], are used to realize the switching function. MTE's having this property are called p-MTE's and networks of p-MTE's are called p-MTE networks.

Any switching function can be realized by a single k-MTE having a sufficiently large k [2-4]. Then it is called a k-multithreshold function (k-MTF).

There are a number of techniques for synthesizing a given switching function by a MTE [3, 4]. However, if for a certain switching function the number of thresholds become very large k, then the practical realization of such a k-MTE often presents serious difficulties. One alternative to this is to realize the switching function by a p-MTE network in which p is smaller than k.

In this paper, a study of p-MTE networks with different modes of interconection has been made in detail. Synthesis procedures are developed for a cascade, a 2-level and a feedforward MTE networks in which each MTE has the identical weight vector

* Department of Computer Science, Yamanashi University, Takeda 4-3-11, Kofu, Japan
** Research Institute of Electrical Communication, Tohoku University, Katahira 2-1-1, Sendai, Japan
*** Research Center of Applied Information Sciences, Tohoku University, Katarira 2-1-1, Sendai, Japan

for the independent (input) variables (**Fig.** 3).

The synthesis of a switching function by loop-free networks of p-MTE's is approached from the point of view of decomposing a single MTE realization of the function into the network. This approach has previously been treated by some researchers [2, 5].

In the present paper, we pay particular attention to the synthesis of networks containing only p-MTE and establish the procedure to minimize the number of such elements which are equivalent to a k-MTE. Our synthesis precedure appears to have some advantages over recent ones [2, 5] from points of view of clearness and generality.

The synthesis procedure for a switching function by cascade p-MTH network in which each MTE may have the different weight vector for the input has also been given. It is shown that this procedure converges for realizing any switching function.

Finally, the upper bound for the maximum number of p-MTE needed to realize an arbitrary n-argument switching function is shown.

II. Definitions and Nations

A k-MTE is specified by an ordered set of weights corresponding to the set of input variables and an ordered set of k-thresholds.

A k-MTE is defined as follows.

$$F(\rho)=Z \quad \text{iff}$$

$$\sum_{i=1}^{n} w_i \cdot x_i(\rho) > T_1$$

$$\text{or} \quad T_{2j} > \sum_{i=1}^{n} w_i \cdot x_i(\rho) > T_{2j+1} \tag{1}$$

and

$$F(\rho)=\bar{Z} \quad \text{otherwise,}$$

where

$F(\rho)$: logical output value of device for vertex ρ, $0 \leq \rho \leq 2^n-1$, $z \in \{0, 1\}$.

$x_i(\rho)$: the i-th logical input value to device for vertex ρ, $x_i(\rho) \in \{0, 1\}$.

w_i : weight of the i-th input, integer.

n : total number of inputs.

T_{2j} : the $2j$-th threshold, a finite real number, $1 \leq j \leq [k/2]$.

The weighted sum

$$\sum_{i=1}^{n} w_i \cdot x_i(\rho)$$

is written as the dot product $\boldsymbol{W} \cdot \boldsymbol{X}(\rho)$.

The ordered set $\{w_1, w_2, \cdots, w_n : T_1, T_2, \cdots, T_k\}$ that specifies a k-MTE will often be denoted by $\{\boldsymbol{W} : \boldsymbol{T}\}$, and called the weight-threshold vector, where $\boldsymbol{W}$ and $\boldsymbol{T}$ is refered to as the weight vector and the threshold vector, respectively.

The definition of k-MTE equivalent to (1) is also written in the following form [4].

$$F(\rho)=Z \quad \text{iff}$$

$$\prod_{j=1}^{k} \{\boldsymbol{W} \cdot \boldsymbol{X}(\rho) - T_j\} > 0$$

and

$$F(\rho)=\bar{Z} \quad \text{iff}$$

$$\prod_{j=1}^{k} \{\boldsymbol{W} \cdot \boldsymbol{X}\{\rho) - T_j\} < 0. \tag{2}$$

A switching function that is realized by a k-MTE is called k-multithreshold function (k-MTF).

A k-MTF can be expressed as following form;

$$F=\{\mathbf{1}[\boldsymbol{A}], \ \wedge[\boldsymbol{W}]\},$$

where

$\boldsymbol{A}=(a_1, \cdots, a_k)$: k dimensional weight vector associated with the constant (inputs) **1** such that $a_{j+1} > a_j$, $1 \leq j \leq k-1$,

and each threshold is given as follows;

$$T_j = \frac{1}{2}\left\{\sum_{i=1}^{n} w_i - a_j\right\}, \quad j=1, \ \cdots, \ k.$$

Definition 1 [5] : A gate type (Z_1, Z_2) of an MTE can be defined for ρ_1 and ρ_2 such that

$$\boldsymbol{W} \cdot \boldsymbol{X}(\rho_1) = \max_{\rho} \boldsymbol{W} \cdot \boldsymbol{X}(\rho) \quad \text{iff} \quad Z_1 = F(\rho_1)$$

and

$$\boldsymbol{W} \cdot \boldsymbol{X}(\rho_2) = \min_{\rho} \boldsymbol{W} \cdot \boldsymbol{X}(\rho) \quad \text{iff} \quad Z_2 = F(\rho_2), \tag{3}$$

where $Z_1, Z_2 \in \{0, 1\}$.

In this case, MTF can be expressed as follows;

$$F=\{\boldsymbol{W} : \boldsymbol{T} : (Z_1, \ Z_2)\}.$$

Definition 2 [5] : Two k-MTE's, $\{\boldsymbol{W}_1 : \boldsymbol{T}_1 : (Z_{11}, Z_{12})\}$ and $\{\boldsymbol{W}_2 : \boldsymbol{T}_2 : (Z_{21}, Z_{22})\}$, are said to be isobaric iff $\boldsymbol{W}_1 = \boldsymbol{W}_2$, $\boldsymbol{T}_1 - \alpha \cdot \boldsymbol{I} = \boldsymbol{T}_2$ and $Z_{11} = Z_{21}$, $Z_{12} = Z_{22}$, where α is a scalar constant, $\boldsymbol{I}$ is the unit vector (k-dimension) and the notation $\boldsymbol{T}_1 - \alpha \cdot \boldsymbol{I}$ represents the vector produced by substructing α to each element of $\boldsymbol{T}_1$.

Definition 3 : If $F_1(\rho)=1$ implies $F_2(\rho)=1$ for all ρ, then $F_1 \Rightarrow F_2$. F_1 and F_2 are comparable if either $F_1 \Rightarrow F_2$ or $F_2 \Rightarrow F_1$.

Definition 4 : For two ordered sets (vectors), $\boldsymbol{V}=(v_1, \cdots, v_m)$ and $\boldsymbol{U}=(u_1, \cdots, u_n)$, $\boldsymbol{V}>\boldsymbol{U}$ iff $v_m > u_1$, where $v_j > v_{j+1}$ and $u_i > u_{i+1}$.

Suppose that we have a standard procedure for numbering the elements in the network ; e.g. we could adopt a left to right convention as in Fig. 3.

Definition 5 : Let i_j be the output of the j-th element in the MTE network of r elements, $i_j=0, 1$. then the ordered set $(i_1, i_2, \cdots, i_r)$ is called "the state sequence" of the network.

Definition 6 : Let the integer $\sum_{j=1}^{r} 2^{r-j} \cdot i_j$ correspond to a state sequence $(i_1, i_2, \cdots, i_r)$.

The distance between two state seduences $(i_1, \cdots, i_r)$ and $(i_1', \cdots, i_r')$ can be defined by

$$\left| \sum_{j=1}^{r} 2^{r-j}(i_j - i_j') \right|.$$

III. Properties of CW p-MTE Networks

Although any arbitrary switching function can be realized by a single MTE, network limitations frequently may make it more desirable to realize the given switching function with a network of MTE's each having the fewer number of thresholds. The procedure for realizing the given swtiching function with a single MTE have been treated by some researchers [3, 4]. Hence, we may decompose the resulting weight-threshold vector into a set of parameters (weight-threshold vectors and weights for functionial inputs) of MTE's in a network.

A consequence of this decomposition is that every MTE in the network has the identical weight vector for the independent (input) variables and the fewer number of thresholds. We call such networks CW MTE networks. In this section, we assume that an appropriate weight vector and k-threshold vector are available and, hence, we are primarily concerned with decompositions.

In CW MTE network of r elements, the points that change the values of the output function H_r (from 0 to 1 or from 1 to 0) as the values of the excitation $\boldsymbol{W} \cdot \boldsymbol{X}$ change, are called "effective threshold (values) in $\boldsymbol{W} \cdot \boldsymbol{X}$," e.g. see Fig. 2. Then we say that the network can have these effective thresholds in $\boldsymbol{W} \cdot \boldsymbol{X}$ at the output element.

Let us now derive some properties of CW MTE networks. Consider the network of two p-MTE's. [See Fig. 1.]. Hence, the output element has as inputs, the complete set of independent variables and the function H_1 which is the output of the first element. ω_{12} is the weight associated with the functional input from the first elememt to the output one as in Fig. 1. Then the output function H_2 can be given by H_1 as follows ;

$$\begin{aligned} H_2 &= \{1[\boldsymbol{A}_2],\ \wedge[\boldsymbol{W}],\ H_1[\omega_{12}]\} \\ &= \overline{H}_1 \cdot \{1[\boldsymbol{A}_2 - \omega_{12} \cdot \boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} \\ &\quad + H_1 \cdot \{1[\boldsymbol{A}_2 + \omega_{12} \cdot \boldsymbol{I}],\ \wedge[\boldsymbol{W}]\}, \end{aligned} \tag{4}$$

where $H_1 = \{1[\boldsymbol{A}_1],\ \wedge[\boldsymbol{W}]\}$,

$$\boldsymbol{T}_1 = \frac{1}{2}\left\{\left(\sum_{i=1}^{n} w/i\right) \cdot \boldsymbol{I} - \boldsymbol{A}_1\right\} \quad \text{and}$$

$$\boldsymbol{T}_2 = \frac{1}{2}\left\{\left(\sum_{i=1}^{n} w/i\right) \cdot \boldsymbol{I} - (\boldsymbol{A}_2 - \omega_{12} \cdot \boldsymbol{I})\right\}.$$

Therefore, each function of (4) can be given as follows.

$$\begin{aligned} &\{1[\boldsymbol{A}_1],\ \wedge[\boldsymbol{W}]\} = \{\boldsymbol{W} : \boldsymbol{T}_1\}, \\ &\{1[\boldsymbol{A}_2 - \omega_{12} \cdot \boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} = \{\boldsymbol{W} : \boldsymbol{T}_2\} \quad \text{and} \\ &\{1[\boldsymbol{A}_2 + \omega_{12} \cdot \boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} = \{\boldsymbol{W} : \boldsymbol{T}_2 - \omega_{12} \cdot \boldsymbol{I}\} \end{aligned} \tag{5}$$

In the following, we restrict ourselves to networks of p-MTE's, where p is odd. Each of the threshold vectors thus has p components, i.e.,

$\boldsymbol{T}_j = \{T_{j1}, T_{j2}, \cdots, T_{jp}\}$, where

$T_{ji} > T_{ji+1}$ for all i.

Each of p-MTE's is assumed to be of the (1, 0) type. Since the functional weight ω_{12} may be either positive or negative, we can show interesting

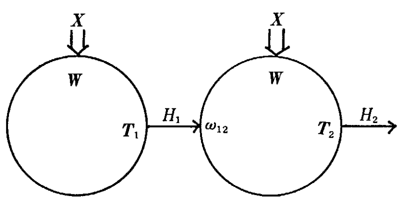


Fig. 1 Two p-MTE network

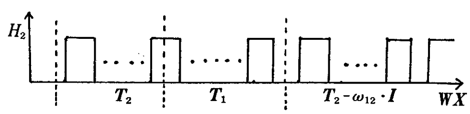


Fig. 2 Output function H_2

consequences on the functional forms of the output as follows ;

(i) Consider ω_{12} to be positive.

If $\boldsymbol{T}_2 > \boldsymbol{T}_2 - \omega_{12}\cdot\boldsymbol{I}$, then

$$\{1[\boldsymbol{A}_2+\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} \supset \{1[\boldsymbol{A}_2-\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\}$$

and from (4),

$$H_2 = \{1[\boldsymbol{A}_2-\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} + \{1[\boldsymbol{A}_1],\ \wedge[\boldsymbol{W}]\}\cdot\{1[\boldsymbol{A}_2+\omega_{12}\cdot\boldsymbol{I}].\ \wedge[\boldsymbol{W}]\}. \tag{6}$$

Then the output MTE have 2p-1 effective thresholds in $W\cdot X$ by properly selecting H_1 and ω_{12}*. [See Foot note1)]

(ii) Consider ω_{12} to be negative.

If $\boldsymbol{T}_2 - \omega_{12} > \boldsymbol{T}_2$, then

$$\{1[\boldsymbol{A}_2-\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} \supset \{1[\boldsymbol{A}_2+\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\}$$

and from (4),

$$H_2 = \overline{\{1[\boldsymbol{A}_1],\ \wedge[\boldsymbol{W}]\}}\cdot\{1[\boldsymbol{A}_2-\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} + \{1[\boldsymbol{A}_2+\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\}, \tag{7}$$

where

$\overline{\{1[\boldsymbol{A}_1],\ \wedge[\boldsymbol{W}]\}} = \{\boldsymbol{W} : \boldsymbol{T}_1 : (0, 1)\}$, $\{1[\boldsymbol{A}_2-\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} = \{\boldsymbol{W} : \boldsymbol{T}_2\}$ and $\{1[\boldsymbol{A}_2+\omega_{12}\cdot\boldsymbol{I}],\ \wedge[\boldsymbol{W}]\} = \{\boldsymbol{W} : \boldsymbol{T}_2-\omega_{12}\cdot\boldsymbol{I}\}$.

If $\boldsymbol{T}_2 < \boldsymbol{T}_1 < \boldsymbol{T}_2 - \omega_{12}\cdot\boldsymbol{I}$, then

$H_2 = \{\boldsymbol{W} : \boldsymbol{T}_2-\omega_{12}\cdot\boldsymbol{I},\ \boldsymbol{T}_1,\ \boldsymbol{T}_2\}$, i.e., the output element can have at most 3p effective thresholds in $W\cdot X$.

We can show the output function H_2 vs. the independent-variables excitation $W\cdot X$ as in Fig. 2.

This points out the advantage of using negative functional weights over positive ones from points of view of the number of effective thresholds. Therefore, in this section, we shall restrict our discussion to negative functional weights.

[A] **Cascade CW p-MTE Network**

Let us consider the cascade CW p-MTE network of k p-MTE's [Fig. 3(a)], in which the i-th MTE has as inputs, the complete set of independent n variables and a function H_{i-1}, where the functional weights $\omega_{j,j+1}$'s are all negative.

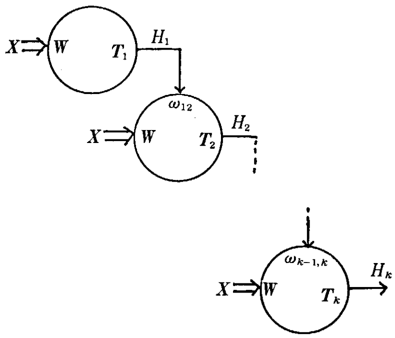
The output of this network, H_k, is given by

$$H_k = \{1[\boldsymbol{A}_k],\ \wedge[\boldsymbol{W}],\ H_{k-1}[\omega_{k-1,k}]\}, \tag{8}$$

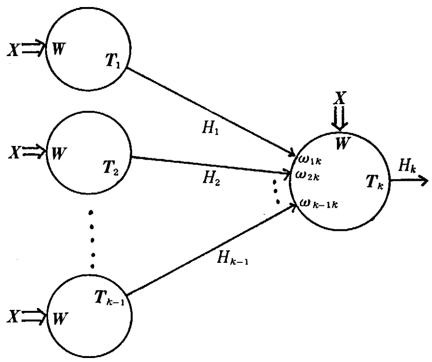
where $H_j = \{1[\boldsymbol{A}_j],\ \wedge[\boldsymbol{W}],\ H_{j-1}[\omega_{j-1,j}]\}$, $1 \leq j \leq K-1$.

A_j : weight vector of element j associated with the constant (inputs) 1.

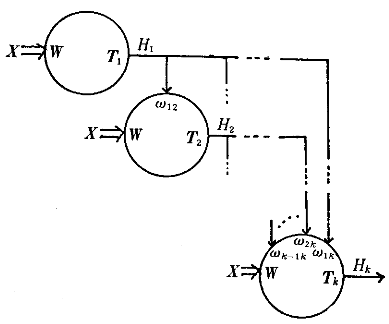
Let us assume the following inequalities between A_j's and $\omega_{j-1,j}$'s,



(a) Cascade CW p-MTE network



(b) 2-level CW p-MTE network



(c) Feedforward CW p-MTE network

Fig. 3

---

* Foot note1)

In the following condition, we can have at most (2p-1) effective thresholds :

Let $h_1 = \{\boldsymbol{W} : T_{12}, \cdots, T_{1p}\}$ and $h_2 = \{\boldsymbol{W} : T_{22}, \cdots, T_{2p}\}$

such that $h_1 \cdot h_2 = 0$.

If $T_{21} - \omega_{12} < T_{1p}$, then $\boldsymbol{T}_2 - \omega_{12}\cdot\boldsymbol{I} < \boldsymbol{T}_1$.

In this case, (2p-1) effective thresholds are $\{T_{12}, \cdots, T_{1p}\}$, $\{T_{22}, \cdots, T_{2p}\}$ and T_{11}(or T_{21}).

$$\begin{aligned}&A_k-\omega_{k-1,k}\cdot I>A_{k-1}-\omega_{k-2,k-1}\cdot I>\cdots>A_2-\omega_{12}\cdot I\\&\quad>A_1>A_2+\omega_{12}\cdot I\\&A_3+\omega_{23}\cdot I>\cdots>A_{k-1}+\omega_{k-2,k-1}\cdot I\\&\quad>A_k+\omega_{k-1,k}\cdot I.\end{aligned}\tag{9}$$

From eq. (9) the following ordered relation can be derived,

$$\begin{aligned}&T_k<T_{k-1}<T_{k-2}<\cdots<T_2<T_1<T_2-\omega_{12}\cdot I<T_3\\&-\omega_{23}\cdot I<\cdots T_{k-1}-\omega_{k-2,k-1}\cdot I<T_k-\omega_{k-1,k}\cdot I,\end{aligned}\tag{10}$$

where

$$T_j=\frac{1}{2}\left\{\left(\sum_{i=1}^{n} w_i\right)\cdot I-(A_j-\omega_{j-1,j}\cdot I)\right\},\ 1\leq j\leq K.$$

The output functional form of H_k can be berived from (9) or (10) because all output functions H_j's of p-MTE' s are comparable : (See Appendix).

i) K is odd,

$$H_k=\sum_{j=1}^{k+1/2}\{F_{2j+1}\cdot\overline{F}_{2j}+G_{2j-1}\cdot\overline{G}_{2j}\}.\tag{11}$$

ii) K is even,

$$H_k=\sum_{j=1}^{k/2}\{F_{2j}\cdot\overline{F}_{2j-1}+G_{2j}\cdot\overline{G}_{2j+1}\},\tag{12}$$

where $F_j=\{1[A_j-\omega_{j-1,j}\cdot I],\ \wedge[W]\}$,

$G_j=\{1[A_j+\omega_{j-1,j}\cdot I],\ \wedge[W]\}$ and $\overline{G}_{k+1}=1$.

Equation (11) is proved in the Appendix. From equation (11) or (12) Theorem 1 can be derived.

Theorem 1 : A cascade CW network of K p-MTE's can have at most $p\cdot(2K\text{-}1)$ effective thresholds in $W\cdot X$ at the output element, if ordered relations (10) are satisfied.

All $p\cdot(2K\text{-}1)$ effective thresholds are not independent, e. g., if all effective thresholds of T_k and T_{k-1} are assigned to some values, then both all one's of $T_k-\omega_{k-1,k}\cdot I$ and $T_{k-1}-\omega_{k-2,k-1}\cdot I$ can not be assigned to arbitrary values, independently. [See Fig. 4].

[B] Two-Level CW p-MTE Network

Let us consider the two-level CW p-MTE network of K p-MTE's [Fig. 3(b)]. In the first level of the network each p-MTE has as inputs, the complete set of independent n-variables, generating the function H_j, $j=1, 2,\cdots, K-1$, and deriving the second level. In the second level single (output) p-MTE has as inputs, the complete set of independent n-variables and the complete set of the function H_j's,| $j=1, 2,\cdots, K-1$, and generates the function H_k.

The output function H_k is given by

$$H_k=\{1[A_k],\ \wedge[W],\ H_1[\omega_{1k}],\ \cdots,\ H_{k-1}[\omega_{k-1,k}]\},\tag{13}$$

where $H_j=\{1[A_j],\ \wedge[W]\},\ 1\leq j\leq K-1.$

Let us assume the following inequalities of A_j's of p-MTE's in the first level,

$$A_{k-1}>A_{k-2}>\cdots>A_1.\tag{14}$$

This implies that

$$H_{k-1}\supset H_{k-2}\supset\cdots\supset H_1.$$

Furthermore, let us assume the following inequalities of parameters,

$$\begin{aligned}&A_k-\Big(\sum_{i=1}^{k-1}\omega_{ik}\Big)\cdot I>A_{k-1}>A_k-\Big(\sum_{i=1}^{k-2}\omega_{ik}-\omega_{k-1,k}\Big)\cdot I\\&>A_{k-2}>\cdots>A_j\\&A_k-\Big(\sum_{i=1}^{j-1}\omega_{ik}-\sum_{i=j}^{k-1}\omega_{ik}\Big)\cdot I>A_{j-1}>\cdots>A_1>A_k\\&+\Big(\sum_{i=1}^{k-1}\omega_{ik}\Big)\cdot I.\end{aligned}\tag{15}$$

From (15) the following ordered relation can be derived,

$$\begin{aligned}&T_k<T_{k-1}<T_k-\omega_{k-1,k}\cdot I<T_{k-2}<\cdots<T_j<T_k\\&-\Big(\sum_{i=1}^{j}\omega_{ik}\Big)\cdot I<T_{j-1}<\cdots T_1<T_k-\Big(\sum_{i=1}^{k-1}\omega_{ik}\Big)\cdot I,\end{aligned}\tag{16}$$

where $T_k=\frac{1}{2}\{(\sum_{i=1}^{n}w_i)\cdot I-(A_k-\sum_{j=1}^{k-1}\omega_{jk}\cdot I)\}$

and $T_j=\frac{1}{2}\{(\sum_{i=1}^{n}w_i)\cdot I-A_j\},\ 1\leq j\leq K-1.$

The output functional form of H_k can be given from (15) and (16) because of the comparability of p-MTF's,

$$H_k=\sum_{j=0}^{k-1}\overline{H}_j\cdot\{1[A_k-\alpha_j\cdot I],\ \wedge[W]\},\tag{17}$$

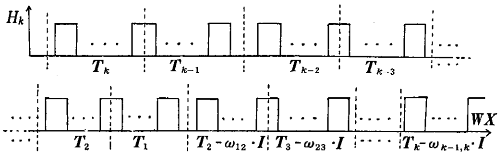


Fig. 4 Output function H_k of cascade CW p-MTE network

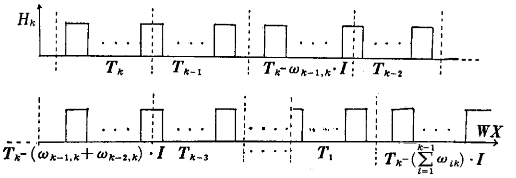


Fig. 5 Output function H_k of 2-level CW p-MTE network

where $\quad \alpha_j = \sum_{i=1}^{j} \omega_{ik} - \sum_{i=j+1}^{k-1} \omega_{ik},$

$H_k = \{\mathbf{1}[\boldsymbol{A}_j], \wedge[\boldsymbol{W}]\}$ and $\overline{H}_0 = 1$. [See Fig. 5].

From equation (17) Theorem 2 can be derived.

Theorem 2 : A two-level CW network of K p-MTE's can have at most $p \cdot (2K-1)$ effective thresholds in $\boldsymbol{W} \cdot X$ at the output element, if ordered relations (16) is satisfied.

All $p \cdot (2K-1)$ effective thresholds are not independent as in the case of the cascade network.

[C] Feedforward CW p-MTE Network

A generalized feedforward network configuration of K CW p-MTE's is shown in Fig. 3(c). The i-th element is a p-MTE having as inputs, the complete set of independent variables and the complete set of H_j's, where H_j is the output function of the j-th element, $j=1, 2, \cdots, i-1$.

First, let us enumerate the effective thresholds realizable by the general feedforward network.

Theorem 3 : A feedforward CW network of K p-MTE's can have at most $p \cdot (2^k - 1)$ effective threholds in $\boldsymbol{W} \cdot X$ at the output element.

Proof : There exist at most 2^r state sequences in the network of r CW p-MTE's. First, let them be ordered such that the distance between each adjacent state sequences is 1 i.e. order from (0, 0, $\cdots$, 0) to (1,1,$\cdots$, 1),so that the dicimal numbers, $\sum_{j=1}^{r} 2^{r-j} \cdot i_j$, corresponding to the state sequences $(i_1, i_2, \cdots, i_r)$ are increased from 0 to $2^r - 1$ as $\boldsymbol{W} \cdot X$ increases, $i_j = 0, 1$.

In general, $(i_1, i_2, \cdots, i_{m-1}, i_m = 0, 1, \cdots, i_r = 1)$ and $(i_1, i_2, \cdots, i_{m-1}, i_m = 1, 0, \cdots, i_r = 0)$ are such adjacent state sequences, where m is the minimum number such that i_m changes from 0 to 1 as $\boldsymbol{W} \cdot X$ increases, $1 \leq m \leq r$. [If $m = r$, then these state sequences are $(i_1, i_2, \cdots, i_{r-1}, i_r = 0)$ and $(i_1, i_2, \cdots, i_{r-1}, i_r = 1)$.].

Next, correspond the p-dimensional vector, $\boldsymbol{T}_m - (\sum_{i \in M} \omega_{im}) \cdot \boldsymbol{I}$, to the pair of these adjacent state sequences, where $\boldsymbol{T}_m$ is the threshold vector of the m-th element and M is the set of i_j's such that $i_j = 1$, $1 \leq j \leq m-1$. [Fig. 6. (a)].

Then it can be proved that there exist following ordered relations of $(2^r - 1)$ p-simensional vectors.

$$(i_1, \cdots\cdots, \underline{i_{m'-1}, i_{m'} = 0, 1, \cdots\cdots, i_{k-1} = 1) \big| (i_1, \cdots\cdots, i_{m'-1}, i_{m'} = 1, 0, \cdots\cdots, i_{k-1}} = 0)$$
$$\boldsymbol{T}_{m'} - (\sum_{i \in M'} \omega_{im'}) \cdot \boldsymbol{I}$$

Fig. 6(**a**) State sequences and corresponding vector

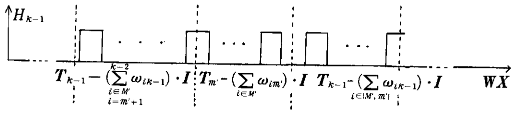


Fig. 6(**b**) Output function H_{k-1} of feedforward CW p-MTE network

$$\boldsymbol{T}_r < \cdots < \boldsymbol{T}_r - (\sum_{\substack{i \in M' \\ i=m'+1}}^{r-1} \omega_{ir}) \cdot \boldsymbol{I} < \boldsymbol{T}_{m'} - (\sum_{i \in M'} \omega_{im'}) \cdot \boldsymbol{I} <$$
$$\boldsymbol{T}_r - (\sum_{i \in \{M', m'\}} \omega_{ir}) \cdot \boldsymbol{I} < \cdots \boldsymbol{T}_r - (\sum_{i=1}^{r-1} \omega_{ir}) \cdot \boldsymbol{I}, \tag{18a}$$

where $\quad 1 \leq m' \leq r-1 \quad$ and $\quad \sum_{\substack{i \in M' \\ i=m'+1}}^{r-1}$

means $\quad \sum_{1 \in M'} + \sum_{i=m'+1}^{r-1}$

The sufficient condition holding the ordered relations (18a) is given as follows.

Assume $\quad \omega_{ip} = \omega_{iq}, \ 1 \leq i \leq r-1, \ 1 \leq p, q \leq r.$

Then, if (18b) holds for each m' $(m' = 1, 2, \cdots, r-1)$, (18a) holds.

$$\boldsymbol{T}_r - (\sum_{i=m'+1}^{r-1} \omega_{ir}) \cdot \boldsymbol{I} < \boldsymbol{T}_{m'} < \boldsymbol{T}_r - \omega_{m'r} \cdot \boldsymbol{I}. \tag{18b}$$

Selecting $\quad \omega_{m'r} < \sum_{i=m'+1}^{r-1} \omega_{ir}, \quad$ (18b) holds.

Therefore (18a) holds.

Next, let us prove that all components of these coroesponding p-dimensional vectors are effective thresholds by the induction on the number of elements.

It is obvious that the theorem holds for $r=1$, since a single p-MTE has p-thresholds.

Assume that all components of $(2^{k-1} - 1)$ p-dimensional vectors are effective thresholds for $r = K-1$. [See Fig. 6(b)]. By assumption we have following relations between state sequences $(i_1, i_2, \cdots, i_{k-1})$, $\boldsymbol{W} \cdot \boldsymbol{X}(\rho)$ and effective thresholds.

$$T_{k-1,p} - (\sum_{\substack{1 \in M' \\ i=m'+1}}^{k-2} \omega_{ik-1}) > \boldsymbol{W} \cdot \boldsymbol{X}(\rho) < T_{k-1,p-1}$$
$$- (\sum_{\substack{i \in M' \\ i=m'+1}}^{k-2} \omega_{ik-1})$$
$$\Rightarrow (i_1, \ i_2, \cdots, i_{m'-1}, i_{m'} = 0, \ 1, \cdots, \ 1, \ i_{k-1} = 1),$$

$$T_{k-1,p-1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})<W\cdot X(\rho)<T_{k-1,p-2}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})$$

$$\Rightarrow (i_1,\ i_2,\cdots,i_{m'-1},\ i_{m'}=0,\ 1,\cdots,1,\ i_{k-1}=0),$$

$$\vdots$$

$$T_{k-1,1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})<W\cdot X(\rho)<T_{m'p}-(\sum_{i\in M'}\omega_{im'})$$

$$\Rightarrow (i_1,\ i_2,\cdots,\ i_{m'-1},\ i_{m'}=0,\ 1,\cdots,1,\ i_{k-1}=1).$$

$$T_{m'p}-(\sum_{i\in M'}\omega_{im'})<W\cdot X(\rho)<T_{m'p-1}-(\sum_{i\in M'}\omega_{im'})$$

$$\Rightarrow (i_1,\ i_2,\cdots,i_{m'-1},\ i_{m'}=1,\ 0,\cdots,0,\ i_{k-1}=0),$$

$$\vdots$$

$$T_{m'1}-(\sum_{i\in M'}\omega_{im'})<W\cdot X(\rho)<T_{k-1,p}-(\sum_{i\in\{M',m'\}}\omega_{ik-1})$$

$$\Rightarrow (i_1,i_2,\cdots,i_{m'-1},i_{m'}=1,0,\cdots,0,i_{k-1}=0).$$

$$T_{k-1,p}-(\sum_{i\in\{M',m'\}}\omega_{ik-1})<W\cdot X(\rho)<T_{k-1,p-1}-(\sum_{i\in\{M',m'\}}\omega_{ik-1})$$

$$\Rightarrow (i_1,i_2,\cdots,i_{m'-1},i_{m'}=1,0,\cdots,0,i_{k-1}=1),$$

$$T_{k-1,p-1}-(\sum_{i\in\{M',m'\}}\omega_{ik-1})<W\cdot X(\rho)<T_{k-1,p-2}-(\sum_{i\in\{M',m'\}}\omega_{ik-1})$$

$$\Rightarrow (i_1,i_2,\cdots,i_{m'-1},i_{m'}=1,0,\cdots,0,i_{k-1}=0).$$

$$\vdots$$

and so on.

In case of $r=K$, we can obtain the following ordered relations of p-dimensional vectors by (18a).

$$\cdots<T_{k-1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})\cdot I<T_k-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik})\cdot I<T_{m'}-(\sum_{i\in M'}\omega_{im'})\cdot I<T_k-(\sum_{i\in\{M',m'\}}\omega_{ik})\cdot I<T_{k-1}-(\sum_{i\in\{M',m'\}}\omega_{ik-1})\cdot I<\cdots. \quad (18c)$$

In this case there exist 2^k-1 ordered p-dimensional vectors in (18c).

Now, let us show that all components of these vectors are effective thresholds.

If $T_{k-1,p}-(\sum_{\substack{i\in M'\\i=m'=1}}^{k-2}\omega_{ik-1})<W\cdot X(\rho)<T_{k-1,p-1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1,\ i_2,\cdots,\ i_{m'-1},\ i_{m'}=0,\ 1,\ \cdots,\ 1,\ i_{k-1}=1)$. Hence $i_k=0$, since

$$T_{kp}>W\cdot X(\rho)+(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik}),\ [\text{from (18c)}].$$

If $T_{k-1,p-1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})<W\cdot X(\rho)<T_{k-1,p-1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1,\ i_2,\cdots i_{m'-1},\ i_{m'}=0,\ 1,\cdots,\ 1,\ i_{k-1}=0)$.

Since following ordered relation is obtained by the procedure of (18a),

$$T_k-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik})\cdot I<T_{k-1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik-1})\cdot I,$$

i.e. $m'=K-1$ in (18a),

we have $T_{k1}<W\cdot X(\rho)+(\sum_{\substack{i\in M'\\i=m'+1}}^{k-2}\omega_{ik})$.

So, $i_k=1$.

$$\vdots$$

If $T_{k-1,1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik-1})<W\cdot X(\rho)<T_{k,p}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1,\ i_2,\cdots,\ i_{m'-1},\ i_{m'}=0,\ 1,\ \cdots,\ 1,\ i_{k-1}=1)$.

Hence $i_k=0$, since

$$T_{k,p}>W\cdot X(\rho)+(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik}),\ [\text{from (18c)}].$$

If $T_{k,p}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik})<W\cdot X(\rho)<T_{k,p-1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1,\ i_2,\cdots,\ i_{m'-1},i_{m'}=0,\ 1,\ \cdots,\ 1,\ i_{k-1}=1)$. Hence $i_k=1$.

$$\vdots$$

If $T_{k,1}-(\sum_{\substack{i\in M'\\i=m'+1}}^{k-1}\omega_{ik})<W\cdot X(\rho)<T_{m',p}-(\sum_{i\in M'}\omega_{im'})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1,\ i_2,\ \cdots,\ i_{m'-1},\ i_{m'}=0,\ 1,\cdots,\ 1,\ i_{k-1}=1)$. Hence $i_k=1$.

If $T_{m',p}-(\sum_{i\in M'}\omega_{im'})<W\cdot X(\rho)<T_{m',p-1}-(\sum_{i\in M'}\omega_{im'})$

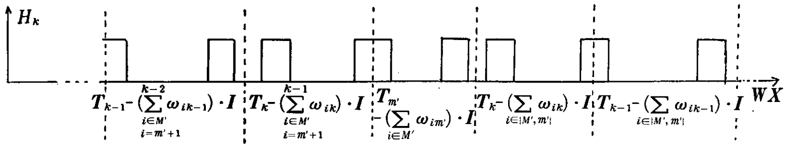


Fig. 7 Output function H_k of feedforword CW p-MTE network

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1, i_2, \cdots, i_{m'-1}, i_{m'}=1, 0, \cdots, 0, i_{k-1}=0)$.

Since following ordered relation is obtained by the procedure of (18a),

$$\boldsymbol{T}_{m'}-(\sum_{i\in M'}\omega_{im'})\cdot\boldsymbol{I}<\boldsymbol{T}_k-(\sum_{i\in\{M',m'\}}\omega_{ik})\cdot\boldsymbol{I},$$

$i_k=0$.

If $\quad T_{m',p-1}-(\sum_{i\in M'}\omega_{im'})<\boldsymbol{W}\cdot\boldsymbol{X}(\rho)<T_{m',p-2}-(\sum_{i\in M'}\omega_{im'})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1, i_2, \cdots, i_{m'-1}, i_{m'}=0, 1, \cdots, 1, i_{k-1}=1)$.

Since $\quad \boldsymbol{T}_k-(\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik})\cdot\boldsymbol{I}<\boldsymbol{T}_{m'}-(\sum_{i\in M'}\omega_{im'})\cdot\boldsymbol{I},$

$i_k=1$.

$\vdots$

If $\quad T_{m',1}-(\sum_{i\in M'}\omega_{im'})<\boldsymbol{W}\cdot\boldsymbol{X}(\rho)<T_{k,p}-(\sum_{i\in\{M',m'\}}\omega_{ik})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1, i_2, \cdots, i_{m'-1}, i_{m'}=1, 0, \cdots, 0, i_{k-1}=0)$. Hence $i_k=0$.

If $\quad T_{k,p}-(\sum_{i\in\{M',m'\}}\omega_{ik})<\boldsymbol{W}\cdot\boldsymbol{X}(\rho)<T_{k,p-1}-(\sum_{i\in\{M',m'\}}\omega_{ik})$

then by the assumption of the induction, the state sequence of $r=K-1$ is $(i_1, i_2, \cdots, i_{m'-1}, i_{m'}=1, 0, \cdots, 0, i_{k-1}=0)$. Hence $i_k=1$.

$\vdots$

If $\quad T_{k,2}-(\sum_{i\in\{M',m'\}}\omega_{ik})<\boldsymbol{W}\cdot\boldsymbol{X}(\rho)<T_{k,1}-(\sum_{i\in\{M',m'\}}\omega_{ik})$

then by the assumption of the induction, the state sequence is $(i_1, i_2, \cdots, i_{m'-1}, i_{m'}=1, 0, \cdots, 0, i_{k-1}=0)$. Hence $i_k=0$.

If $\quad T_{k,1}-(\sum_{i\in\{M',m'\}}\omega_{ik})<\boldsymbol{W}\cdot\boldsymbol{X}(\rho)<T_{k-1,}-(\sum_{i\in\{M',m'\}}\omega_{i,k-1})$

then by the assumption of the induction, the state sequence is $(i_1, i_2, \cdots, i_{m'-1}, i_{m'}=1, 0, \cdots, 0, i_{k-1}=0)$. Hence $i_k=1$.

It can be also proved that all components of $\boldsymbol{T}_{k-1}-(\sum_{i\in\{M',m'\}}\omega_{ik-1})\cdot\boldsymbol{I}$ are effective thresholds.

Therefore, it has been proved that for any $m'(1\leq m'\leq K-2)$ all components of p-dimensional vectors in (18c) are effective thresholds. [See Fig. 7].

Since, in case of $r=K$, the number of these p-dimensional vectors is 2^k-1, the total number of effective thresholds is $p\cdot(2^k-1)$.

This completes the proof. (Q.E.D.)

Example : Construct effective thresholds for the network of three p-MTE's.

In the case of two p-MTE's, the effective threshold vectors have been given as follows :

$[\boldsymbol{T}_2, \boldsymbol{T}_1, \boldsymbol{T}_2-\omega_{12}\cdot\boldsymbol{I}]$. [See Fig. 2].

The following ordered effective threshold vectors are constructed by the procedure mentioned above.

$[\boldsymbol{T}_3, \boldsymbol{T}_2, \boldsymbol{T}_3-\omega_{23}\cdot\boldsymbol{I}, \boldsymbol{T}_1, \boldsymbol{T}_3-\omega_{13}\cdot\boldsymbol{I}, \boldsymbol{T}_2-\omega_{12}\cdot\boldsymbol{I}, \boldsymbol{T}_3-(\omega_{13}+\omega_{23})\cdot\boldsymbol{I}]$.

In the case of four p-MTE's, ordered effective threshold vectors are given as follows.

$[\boldsymbol{T}_4, \boldsymbol{T}_3, \boldsymbol{T}_4-\omega_{34}\cdot\boldsymbol{I}, \boldsymbol{T}_2, \boldsymbol{T}_4-\omega_{24}\cdot\boldsymbol{I}, \boldsymbol{T}_3-\omega_{23}\cdot\boldsymbol{I}, \boldsymbol{T}_4-(\omega_{24}+\omega_{34})\cdot\boldsymbol{I}, \boldsymbol{T}_1, \boldsymbol{T}_4-\omega_{14}\cdot\boldsymbol{I}, \boldsymbol{T}_3-\omega_{13}\cdot\boldsymbol{I}, \boldsymbol{T}_4-(\omega_{14}+\omega_{34})\cdot\boldsymbol{I}, \boldsymbol{T}_2-\omega_{12}\cdot\boldsymbol{I}, \boldsymbol{T}_4-(\omega_{14}+\omega_{24})\cdot\boldsymbol{I}, \boldsymbol{T}_3-(\omega_{13}+\omega_{23})\cdot\boldsymbol{I}, \boldsymbol{T}_4-(\omega_{14}+\omega_{24}+\omega_{34})\cdot\boldsymbol{I}]$

The next step is to consider the decrease of the number of effective thresholds when the ordered relation of (18c) does not hold, where $m'=1, 2 \cdots, K-1$. Two cases are considered as follows :

1) $\boldsymbol{T}_{m'}-(\sum_{i\in M'}\omega_{im'})\cdot\boldsymbol{I}<\boldsymbol{T}_k-(\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik})\cdot\boldsymbol{I}<\boldsymbol{T}_k-(\sum_{i\in\{M',m'\}}\omega_{ik})\cdot\boldsymbol{I}$.

If $\quad \boldsymbol{W}\cdot\boldsymbol{X}(\rho)<T_{m'p}-\sum_{i\in M'}\omega_{im'}$, then the state sequences is $(i_1, \cdots, i_{m'-1}, i_{m'}=0, 1, \cdots, 1, i_k=0)$, if $T_{m'p}-\sum_{i\in M'}\omega_{im'}<\boldsymbol{W}\cdot\boldsymbol{X}(\rho)<T_{m'p-1}-\sum_{i\in M'}\omega_{im'}$, then the state sequence is $(i_1, \cdots, i_{m'-1}, i_{m'}=1, 0, \cdots, 0, i_k=0)$ and so on. Therefore, all components of $\boldsymbol{T}_{m'}-(\sum_{i\in M'}\omega_{im'})\cdot\boldsymbol{I}$ are not effective

thresholds.

If $T_{m'1}-\sum_{i\in M'}\omega_{im'}<W\cdot X(\rho)<T_{kp}-\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik}$,

then the state sequence is $(i_1,\ \cdots,\ i_{m'-1},\ i_{m'}=1,\ 0,\ \cdots,\ 0,\ i_k=0)$,

if $T_{ik}-\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik}<W\cdot X(\rho)<T_{kp-1}-\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik}$,

then the state sequence is $(i_1,\ \cdots,\ i_{m'-1},\ i_{m'}=1,\ 0,\ \cdots,\ 0,\ i_k=0)$, since

$W\cdot X(\rho)<T_{kp}-\sum_{i\in\{M',m\}}\omega_{ik}$, and so on.

Therefore, all components of $T_k-(\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik})\cdot I$ are not effective thresholds.

In particular, if $\omega_{ip}=\omega_{iq}$, $1\le i\le K-1$, $1\le p,\ q\le K$, then $T_{m'}<T_k-(\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik})\cdot I$.

Since this relation holds independently of the set M' and the number of threshold vectors has been 2^{m-1}, where $m=1,\ 2,\ \cdots,\ K$, the number of effective thresholds turns out to be

$$p\cdot\{(2^k-1)-2\cdot 2^{m-1}\}.$$

Denoting by J the set of such m's the number of effective thresholds is given as follows.

$$p\cdot\{(2^k-1)-2\sum_{m\in J}2^{m-1}\}. \tag{19}$$

2) $T_k-(\sum_{\substack{i\in M'\\ i=m'+1}}^{k-1}\omega_{ik})\cdot I<T_k-(\sum_{i\in\{M',m'\}}\omega_{ik})\cdot I$

$<T_{m'}-(\sum_{i\in M'}\omega_{im'})\cdot I$.

Then it can also be proved as in 1) that all components of $T_k-(\sum_{i\in\{M',m'\}}\omega_{ik})\cdot I$ and $T_{m'}-(\sum_{i\in M'}\omega_{im'})\cdot I$ are not effective thresholds.

The number of effective thresholds is same in the case of 1). From two cases, the following theorem is derived.

Theorem 4 : If the thresholds of each elements and weights for the functional inputs are varied in feedforward CW p-MTE network, the number of generated effective thresholds can be any form

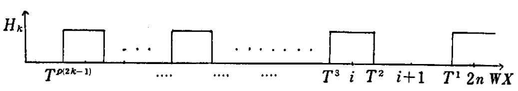


Fig. 8 Aspect of output function of cascade (or 2-level) CW p-MTE network

of $p\cdot(2s-1)$, where $s=1,\ 2,\ \cdots,\ 2^{k-1}$, and K is the number of elements.

IV. Capability of CW p-MTE Networks

This section considers the relation between the number of elements K of a CW p-MTE network and the number of synthesized n-argument switching functions, assuming that the thresholds and weights for the functional inputs of all elements are variable.

Let the weight vector be called the canonical weight vector such that the weight for x_i is 2^{i-1}, $i=1,\ \cdots,\ n$.

Lemma 1[10] : The number of different functions realized by changing the threshold of a single threshold element (1-MTE) but keeping the weights of the n variable inputs fixed is not greater than 2^n+1, including the two functions that are identically 1 and 0.

The following theorems can be drived by the definition of the canonical weight vector and Lemma 1.

Theorem 5 : If the threshold vectors and the weights for the functional inputs of p-MTE's in cascade or 2-level CW p-MTE network are varied, then the number $R_{k(n)}$ of n-argument switching functions synthesized is given as follows :

$$R_{k(n)}\le 2^n+1+\sum_{i=1}^{2n-1}\sum_{j=1}^{p(2k-1)-1/2}\binom{i}{2j-1}\cdot(2^n-i), \tag{20}$$

where K is the number of elements.

Proof : When the value of $W\cdot X(\rho)$ is different for every ρ, for example by using canonical weight vector, these values can be enumerated as $1,\ 2,\cdots,\ 2^n$ in increasing order in $W\cdot X$.

In a cascade or a two-level network of K CW p-MTE's, the same maximum number of effective thresholds appears, and these can also be enumerated as in Eig. 8.

Assume that T^2 is fixed such that $i\le T^2\le i+1$ and every T^i, such that $T^i\le T^2$, is fixed and varying T^1, then we obtain (2^n-i) pattern of H_k. Fixing T^1 and T^2, and varying $(2j-1)$ effective thresholds $T^3,\ T^4,\ \cdots,\ T^{2j+1}$, we obtain $\binom{i}{2j-1}$

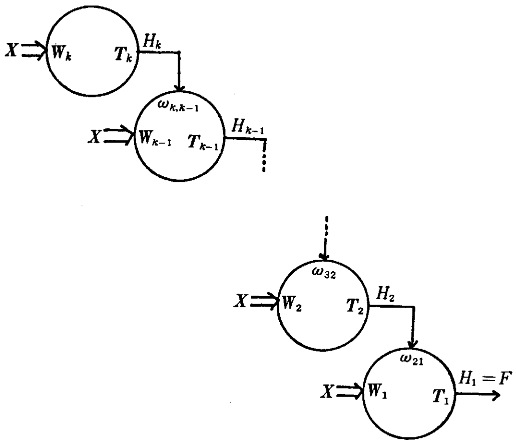


Fig. 9 Cascade p-MTE network

patterns of H_k. Therefore, $\binom{i}{2j-1}\cdot(2^n-i)$ different patterns of H_k are obtained for some i, where $1\leq j\leq[i+1/2]$ and also $1\leq j\leq p\cdot(2k-1)-1/2$.

Since each pattern corresponds to a switching, function, the number of pattern obtained is equal to the number of switching functions. Since all generated effective thresholds are not independent, the upper bound for $R_{k(n)}$ is only given (if $p=1$, then the equality holds [6]). (Q.E.D.)

Theorem 6 : If the threshold vectors and the weights for the functional input of p-MTE's in feedforward CW p-MTE network are varied, then the number $R_{k(n)}'$ of n-argument switching functions synthesized is given as follows :

$$R_{k(n)}'\leq 2^n+1+\sum_{i=1}^{2n-1}\sum_{j=1}^{p(2^k-1)-1/2}\binom{i}{2j-1}\cdot(2^n-i), \tag{21}$$

where K is the number of elements.

This theorem can be proved in the same way as Theorem 5 using the fact that the maximum number of effective thresholds in the feedforward network of K CW p-MTE's is at most $p\cdot(2^k-1)$.

V. Synthesis of Cascade p-MTE Network

In this section we show an algorithm for realizing the given switching function by a cascade p-MTE network, where each MTE may have the different weight vector. In the algorithm for the synthesis the resulting network is of the form shown in Fig. 9. The $(i+1)$-th MTE is connected to the adjacent i-th MTE positively, i.e. $\omega_{i+1,i}>0$ for all i, and each MTE has p-thresholds (suppose p is odd and $p\geq 3$). In the following algorithm suppose that each weight vector of MTE's is obtained by some procedures, (e.g. [3]).

It can be shown that the algorithm converges for realizing arbitrary switching functions.

[Algorithm]

Given an arbitrary n-argument switching function F and suppose that p is odd, $p\geq 3$ and $Z=1$ in eq. (2).

Let H_i be the output function of the i-th element, e.g. $F=H_1$.

Let

$$f^i(\rho)=\prod_{j=1}^{p}\{W_i\cdot X(\rho)-T_{ij}\} \tag{22}$$

and

$$f_i(\rho)=\prod_{j=1}^{p}\{W_i\cdot X(\rho)-T_{ij}+\omega_{i+1,i}\cdot H_{i+1}(\rho)\}, \tag{23}$$

where $0\leq\rho\leq 2^n-1$ and W_i is the weight vector associated with the i-th element.

STEP 1 : Suppose that $F=\{w_{11},\cdots,w_{1n}:T_{11},\cdots,T_{1r}\}=\{W_1:T\}$. (In the following suppose each weight vector is determined by [3]).

If $r\leq p$ then F can be realized by a single p-MTE, else go to STEP 2,

STEP 2 : The threshold vector of the i-th element, $i=1, 2, \cdots$, is determined as follows :

First, p-MTF H^i is defined as follows.

$$f^i(\rho)>0\Longleftrightarrow H^i(\rho)=1,\ f^i(\rho)<0\Longleftrightarrow H^i(\rho)=0.$$

Then, the threshold vector $(T_{i1},\cdots,T_{ip})$ of the i-th element is chosen so that $H_i\supset H^{i}$**. [See Foot note 2]

If $H^i=H_i$ then F can be realized by cascade p-MTE network consisting of i-elements, else go to STEP 3, (i.e. in the case of which H_i is not realized by a p-MTF). STEP 1 is the particular case in STEP 2.

STEP 3 : In equation (23), the output function

** Foot note 2)

i.e. the case that $f^i(\rho)>0$ ($\Longleftrightarrow H^i(\rho)=1$) and $H_i(\rho)=0$ is forbidden.

For example, if H_i is q-MTF such that $q>p$ and is realized by $\{W_i;T_{i1},\cdots,T_{iq}\}$, then the threshold vector of H^i is chosen T_{i1} and any other $p-1$ thresholds $(T_{i2j},T_{i2j+1},T_{i2l},T_{i2l+1},\cdots)$ from q thresholds.

H_{i+1} of the $(i+1)$-th element and weight for the functional input $\omega_{i+1,i}$ are determined as follows:

1) $f_i(\rho)<0$ and $H_i(\rho)=1 \Rightarrow H_{i+1}(\rho)=1$,

2) $f^i(\rho)>0$ and $H_i(\rho)=1 \Rightarrow H_{i+1}(\rho)=\phi$,

3) $f^i(\rho)<0$ and $H_i(\rho)=0=H_{i+1}(\rho)=0$,

where ϕ is 0 or 1.

$\omega_{i+1,i}$ must be chosen such that

$$f_i(\rho)>0 \Longleftrightarrow H_i(\rho)=1,\ f_i(\rho)<0 \Longleftrightarrow H_i(\rho)=0,$$

Let $\omega_{i+1,i}=T_{i1}-\min_{\{f^i(\rho)<0,\ H_i(\rho)=1\}} W_i \cdot X(\rho)+\delta \quad (\delta>0)$,

then this condition is satisfied.

Then change i to $i+1$ and go to STEP 2.

It can be proved that the algorithm converges for realizing any switching function as follows.

In 2) of STEP 3, let $\phi=0$ for at least one ρ, then $H_i \supset H_{i+1}$.

Therefore, there is K such that $F=H_1 \supset H_2 \supset \cdots \supset H_k$ and H_k is realized by a single p-MTE. Hence, the algorithm mentioned above converges for realizing any switching function.

Example: $F(x_1, x_2, x_3, x_4)=\sum(0, 2, 5, 11, 12, 14)$.

Suppose $p=3$ and each weight-threshod vector is determined by [3].

STEP 1 and STEP 2:

$$F=\{-2, -1, 3, -4,\ : 2.5, 0.5, -0.5, -2.5, -3.5, -4.5, -5.5\}.$$

Let $\boldsymbol{W}_1=(-2, -1, 3, -4)$ and $\boldsymbol{T}_1=(2.5, -4.5, -5.5)$.

STEP 3:

$$H_2=\sum(0, 2, 11, 12, 14)$$

$$\omega_{21}=T_{i1}-\min_{\{f'(\rho)<0,\ H_1(\rho)=1\}} W \cdot X(\rho)+\delta$$

$$=2.5-(-3)+\delta=6.5, \quad (\text{let } \delta=1).$$

$$H_2=\{3, 4, -1, -2;\ 4.5, 0.5, -0.5\}.$$

Hence, H_2 can be realized by a three-MTE.

Therefore, F is realized by the cascade network of two 3-MTE's.

Theorem 7: In the algorithm mentioned above, let K be the number of p-MTE needed for synthesizing all n-argument switching functions, then the upper bound for K is given as follows,

$$K \leq [(2^n-2)/(p-1)], \tag{24}$$

where $p \geq 3$, $[Y]$: Gaussian notation.

Proof: It is sufficient to consider the number of ρ's less than 2^{n-1} such that $F(\rho)=1$.

Given n-argument switching function F, we can realize F by some q-MTE, of which parameters are $\{\boldsymbol{W}_1;\ T_1, T_2, \cdots, T_q\}$, (e.g. by using procedure [3]).

If $p \geq q$, then F can be realized by a p-MTE.

If $p<q$, then let each weight vector $\boldsymbol{W}_i$ of p-MTE's be $\boldsymbol{W}_1$.

Then each threshold vector $\boldsymbol{T}_i$ of p-MTE' s is determined according to the threshold vector $(T_1, T_2, \cdots, T_q)$ as follows:

$$\boldsymbol{T}_1=(T_1, \cdots, T_p), \boldsymbol{T}_2=(T_1, T_{p+1}, \cdots, T_{2p-1}), \cdots,$$

$$\boldsymbol{T}_i=(T_1, T_{(i-1)\cdot(p-1)+2}, \cdots, T_{i(p-1)+2}), \cdots,$$

$$2 \leq i \leq K.$$

The assignment of threshold vectors mentioned above and the fact that $q \leq 2^n-1$ determine the number of p-MTE, K, needed for synthesizing arbitrary n-argument switching functions as follows:

$$(p-1)\cdot(K-1)+p \leq 2^n-1.$$

Therefore, eq. (24) is derived.

(Q.E.D.)

VI. Conclusion

In this paper, a systematic study of p-MTE networks has been made.

First, we have given the properties, construction methods and capabilities of the cascade, the 2-level and the feedforward CW p-MTE networks. The most generalized loop-free feedforward CW p-MTE network has been explored in detail.

Next, we have given an algorithm for synthesizing the cascade p-MTE network realization of an arbitrary switching, function where each MTE of the network may have the different weight vector for the inputs.

Finally, the upper bound for the maximum number of p-MTE needed in the algorithm to realize an arbitrary n-argument switching function also has been given.

ACKNOWLEDGEMENT

The authors wish to thank the colleagues for their valuable discussions on this work. We also want to thank Miss Mutsuho Hayasaka for typing the manuscript.

APPENDIX

Proof for eq. (12) by induction.

Induction base : trivial.

Suppose that eq. (11) holds.

Then in the case of $(k+1)$ element, the following equation holds.

$$
\begin{aligned}
H_{k+1} &= H_k\cdot\{1[A_{k+1}+\omega_{k,k+1}\cdot I], \wedge[W]\} \\
&\quad +\overline{H}_k\cdot\{1[A_{k+1}-\omega_{k,k+1}\cdot I], \wedge[W]\} \\
&= H_k\cdot G_{k+1}+\overline{H}_k\cdot F_{k+1} \qquad \text{(A1)}
\end{aligned}
$$

According to the ordering between parameters in eq. (9) the following inequalitieshold :

$$A_{k+1}-\omega_{k,k+1}\cdot I > A_k-\omega_{k-1,k}\cdot I \quad \text{and}$$

$$A_k+\omega_{k-1,k}\cdot I > A_{k+1}+\omega_{k,k+1}\cdot I.$$

Hence, $\quad F_{k+1}\supset F_k\supset\cdots\supset\cdots\supset G_k\supset G_{k+1}$.

Therefore, eq, (A1) become,

$$
\begin{aligned}
H_{k+1} &= (F_k\cdot\overline{F}_{k-1}+\cdots+F_1\cdot\overline{G}_2+\cdots+G_3\cdot\overline{G}_4+\cdots \\
&\quad +G_{k-2}\cdot\overline{G}_{k-1}+G_k)\cdot G_{k+1}+(\overline{F}_k+F_{k-1}\cdot\overline{F}_{k-2} \\
&\quad +\cdots+F_2\cdot F_1+G_2\cdot\overline{G}_3+\cdots+G_{k-1}\cdot\overline{G}_k)\cdot F_{k+1} \\
&= G_{k+1}+F_{k+1}\cdot\overline{F}_k+\cdots+F_2\cdot\overline{F}_1+G_2\cdot\overline{G}_3 \\
&\quad +\cdots+G_{k-1}\cdot\overline{G}_k \\
&= \sum_{j=1}^{k+1/2}\{F_{2j}\cdot\overline{F}_{2j-1}+G_{2j}\cdot\overline{G}_{2j+1}\}, \qquad \text{(A2)}
\end{aligned}
$$

where $\quad \overline{G}_{k+2}=1$.

Therefore, eq. (A 2) equals to the case of $(k+1)$ in eq. (12).

The proof for the case that $(k+1)$ is odd is similar one. (Q.E.D.)

REFERENCES

[1] P. Ercoli and L. Mercurio, "Theshold logic one or more than one threshold". Proc. 1962 IFIPS Congr., pp 741—745, 1962.

[2] D. R. Haring, "Multithreshold threshold elements," IEEE Trans. Electron. Comput., vol. EC-15, pp 45—65, Feb. 1966.

[3] D.R. Haring and R.J. Diehuis, "A realization procedure for multithreshold threshold elements", IEEE Trans. Eelctron. Comput., vol. EC-16, pp 828—835, June 1967.

[4] C. W. Mow and K.S.Fu, "Loop-free threshold element structures", IEEE Trans. Comput., vol. C-18, March 1969.

[5] S. Ghosh and A. K. Choudhury, "Cascaded multithreshold networks", IEEE Trans. Comput., Vol. C-20, pp 655—662, June 1971.

[6] A. Imamiya, S. Noguchi and J. Oizumi, "Iterative threshold logic networks", 4-th Hawaii Intern. Conf. on Sys. Sce. (HICSS), 1971.

[7] ———, ———, ———, "Multithreshold networks", 5-th HICSS 1972.

[8] ———, ———, ———, "Ternary threshold-logic networks realizing 3-valued multithreshold function", 6-th HICSS 1973.

[9] ———, ———, ———, "On multithreshold function and multithreshold element networks", 7-th HICSS 1974.

[10] S. Muroga, "Threshold logic and its applications", *John Wiley&Sons,* 1971.