

A Note on Dynamics of a Swing

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Synopsis

Sometimes the motion of a swing is explained as an example of parametric excitation phenomena where the spring stiffness in a vibrating system undergoes a periodic change with the frequency twice as much as the natural frequency of the system.

This explanation is wrong, or at least insufficient in two points. The one point is that the parametric excitation in the case of swing motion is due not only to the periodic change of spring stiffness, but also to that of damping coefficient and the positive effect of the latter predominates over the negative one of the former. And the other point is that the motion of the driving person is usually with the frequency not twice as much as, but equal to the natural frequency of a pendulum composed of the swing and the driving person.

These two points are clarified in this paper by considering the energy balance of swing driving with results retaining the essential features of swing motion as to the fundamental harmonic of the excited oscillation and within errors of small quantity of the 1st order.

1. Introduction

Sometimes the motion of a swing is explained as an example of parametric excitation phenomena where the spring stiffness in a vibrating system undergoes a periodic change with the frequency twice as much as the natural frequency of the system. The spring stiffness of a pendulum composed of a swing and a driving person sitting on it is equal to mg/l , m being the mass, g being the acceleration of gravity and l being the equivalent length of the pendulum. The driving person changes the equivalent length l periodically by making up and down the center of gravity of the pendulum with repeatedly bending and stretching his knees.

But, this explanation is wrong, or at least insufficient, in two points, as understood from the following considerations and as seen from simple observations of actual swing motions.

The one point is that the parametric excitation in the case of the motion of a swing is due not only to the periodic change of the spring stiffness but also

to that of the damping coefficient and the positive effect of the latter predominates over the negative one of the former. And the other point is that the motion of the driving person relative to the swing is usually with the frequency not twice as much as, but equal to the natural frequency of a pendulum composed of the swing and the driving person. This circumstance is easily observed on playgrounds of our nearby kindergartens or city parks.

2. Periodic change of spring stiffness and damping coefficient

We consider a simplified model of a swing with a driving person which is expected to retain the essential features as a dynamical system. It is illustrated in Fig. 1. A massless rod of length L with a step at the lower end can rotate freely about a horizontal sustaining axis at the upper end. An imaginary creature regarded as a mass point holding the rod with its massless hand is capable of sliding along the rod to adjust its

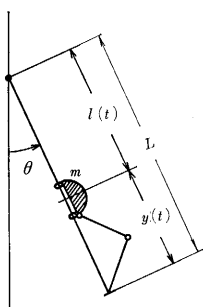


Fig. 1 Simplified model of a swing with a driving person

distance from the step according to any prescribed mode by repeatedly bending and stretching its knees.

We denote the mass of the creature by m , the distances of its center of gravity from the sustaining axis and from the step by $l(t)$ and $y(t)$ respectively, so that

$$y(t) = L - l(t) \quad (1)$$

We regard oscillations of this system as those of a pendulum of variable length $l(t)$. The Newton's principle for a rotating motion about a fixed axis says that the time rate of change of the moment of momentum is equal to the moment of the applied force about the axis, namely in our case that

$$d/dt(ml^2\dot{\theta}) = -mgl \sin \theta \quad (2)$$

where θ is the angle of inclination of the rod measured from the vertical and g is the acceleration of gravity. Performing differentiation and dividing by ml^2 , we obtain

$$\ddot{\theta} + 2(\dot{l}/l)\dot{\theta} + (g/l) \sin \theta = 0 \quad (3)$$

We assume that the variation of the distance $l(t)$ is periodic and is a small quantity of the first order as represented by

$$l(t) = l_0(1 - \varepsilon f(\tau)) \quad (4)$$

where l_0 is the mean distance of the mass m from the sustaining axis, ε is a small constant,

$$\tau = \omega_0 t \text{ and } \omega_0 = \sqrt{g/l_0} \quad (5)$$

are the nondimensional time and the natural circular frequency of a pendulum of length equal to the mean distance l_0 .

Changing the independent variable from the ordinary time t to the nondimensional time τ , we obtain as an equation of motion for small oscillations

$$\theta'' - 2\varepsilon f'(\tau)\theta' + (1 + \varepsilon f(\tau))\theta = 0 \quad (6)$$

where small quantities over the 2nd power of ε and θ are neglected and the prime ' means differentiation with respect to τ . The coefficient of θ and that of θ' correspond to the spring stiffness and the damping coefficient respectively.

As widely known, we can excite a steadily growing oscillation by periodically varying the spring stiffness k and the damping coefficient c in a vibrating system whose equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (7)$$

The favorite modes of varying them for excitation are different between two parameters k and c . This circumstance is most easily understood by the following considerations. We denote the mean value of k by k_0 and assume a mode of variation of k with a symmetric square wave form as illustrated in Fig. 2. The phase relation may be adjusted as follows. The spring has a smaller stiffness $k_0 - \Delta k$ during the interval where the mass m is receding from the equilibrium point which is taken as the origin of the coordinate of displacement x , and has a greater stiffness $k_0 + \Delta k$ during the interval where the mass is approaching to the origin. Thus the mass acquires a greater velocity than that at the start after one excursion from and back to the origin, so that the oscillation grows up. After one excursion the kinetic energy is increased by the amount

$$(1/2)(k_0 + \Delta k)a^2 - (1/2)(k_0 - \Delta k)a^2 = (\Delta k)a^2 \quad (8)$$

where a is the slowly varying amplitude of vibration. This quantity divided by the half period

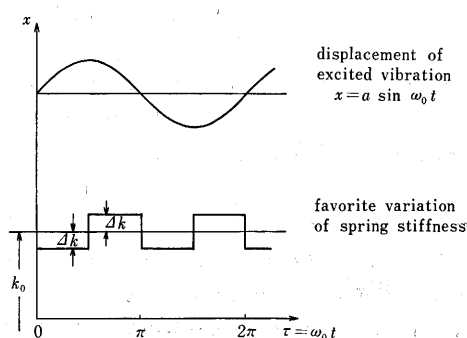


Fig. 2 Favorite variation of spring stiffness with square wave form

π/ω_0 gives the mean time rate of increase of the total vibration energy $(1/2)ma^2\omega_0^2$, so that

$$d/dt((1/2)ma^2\omega_0^2) = (\Delta k)a^2/(\pi/\omega_0), \quad \omega_0 \equiv \sqrt{k_0/m} \quad (9)$$

namely

$$\dot{a}/a = (\omega_0/\pi)\Delta k/k \quad (10)$$

As for the variation of the damping coefficient c , the favorite mode of variation is easily seen to be that of smaller or negative damping at the greater velocities and of greater or positive damping at the smaller ones. Assuming also a mode of variation with symmetric square wave form with the mean value c_0 and the amplitude Δc and taking the phase relations as illustrated in Fig. 3, the increase during one period of the total vibration energy is calculated as

$$\begin{aligned} \int c \dot{x}^2 dt &= \omega_0(-c_0 + \Delta c) \int_{-\pi/4}^{\pi/4} a^2 \cos^2 \tau d\tau \\ &\quad - \omega_0(c_0 - \Delta c) \int_{\pi/4}^{3\pi/4} a^2 \cos^2 \tau d\tau \\ &= \omega_0(\Delta c)a^2 - (1/2)c_0a^2\omega_0 \end{aligned} \quad (11)$$

from which, by similar procedures to the case of variation of k , we obtain

$$\dot{a}/a = \Delta c/(m\pi) - \zeta_0\omega_0 \quad (12)$$

ζ_0 being the damping ratio of the mean damping coefficient c_0 , namely

$$\zeta_0 = c_0/(2\sqrt{mk_0}) \quad (13)$$

We will next investigate in general the effect of periodic variations of k and c of arbitrary wave

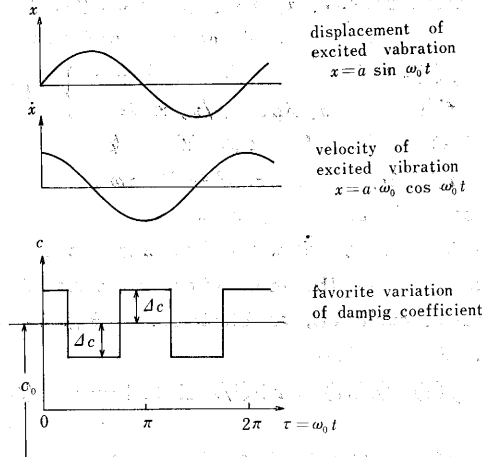


Fig. 3 Favorite variation of damping coefficient with square wave form

forms, which are represented by

$$\begin{aligned} c &= c_0 - \delta c = \sqrt{k_0 m}(2\zeta_0 - \varepsilon g(\tau)) \\ k &= k_0 - \delta k = k_0(1 - \varepsilon h(\tau)) \end{aligned} \quad (14)$$

Then, the eq. (7) can be transformed into

$$m\ddot{x} + k_0x = -c_0\dot{x} + ((\delta c)\dot{x} + (\delta k)x) \quad (15)$$

and, multiplying this equation by x , we obtain

$$d((1/2)(m\dot{x}^2 + k_0x^2)) = (-c_0\dot{x}^2 + (\delta c)\dot{x}^2 + (\delta k)x\dot{x})dt \quad (16)$$

As our excited vibration is of slowly varying amplitude $a(\tau)$ and is represented by

$$x = a(\tau) \sin \tau, \quad \tau = \omega_0 t, \quad \omega_0 = \sqrt{k_0/m} \quad (17)$$

we obtain from similar considerations to the case of square wave form

$$a'/a = (1/2\pi) \int_0^{2\pi} (g(\tau) \cos^2 \tau + h(\tau) \sin \tau \cos \tau) d\tau - \zeta_0 \quad (18)$$

The fundamental harmonic and the higher harmonics other than the 2nd harmonic of the periodic function $g(\tau)$ and $h(\tau)$ contribute nothing to the integral in eq. (18). Only the term of the form $A \cos 2\tau$ in $g(\tau)$ and the term of the form $B \sin 2\tau$ in $h(\tau)$ are effective and give

$$a'/a = (\varepsilon/4)(A+B) - \zeta_0, \quad (19)$$

A being the amplitude of the cosine part of the 2nd harmonic in $g(\tau)$ and B being that of the sine part of the 2nd harmonic in $h(\tau)$.

The former results eq. (10) eq. (12) correspond to the case $\zeta_0 = 0$, $A = 0$, $B = (4/\pi)\Delta k/k_0$ and to the case $\zeta_0 \neq 0$, $A = (4/\pi)\Delta c/\sqrt{mk_0}$, $B = 0$ respectively.

Assuming the variations of sinusoidal wave form (Fig. 4) instead of those of square wave form, we have

$$a'/a = (1/4)(\Delta c/\sqrt{mk_0} + \Delta k/k_0) - \zeta_0 \quad (20)$$

Δc and Δk being the amplitudes of variations of c and k respectively.

Lastly, our equation (6) for the motion of a swing gives

$$h(\tau) = -f(\tau), \quad g(\tau) = 2f'(\tau), \quad \zeta_0 = 0 \quad (21)$$

and for $f(\tau) = \sin 2\tau$ we have $A = 4$ and $B = -1$, so that

$$a'/a = 3\varepsilon/4 \quad \text{and} \quad \theta = a_0 e^{(3\varepsilon/4)\omega_0 t} \sin \omega_0 t \quad (22)$$

a_0 being the initial amplitude of swing oscillation.

This mode of excitation is illustrated in Fig. 5a and fig. 5b. The positive effect of the periodic

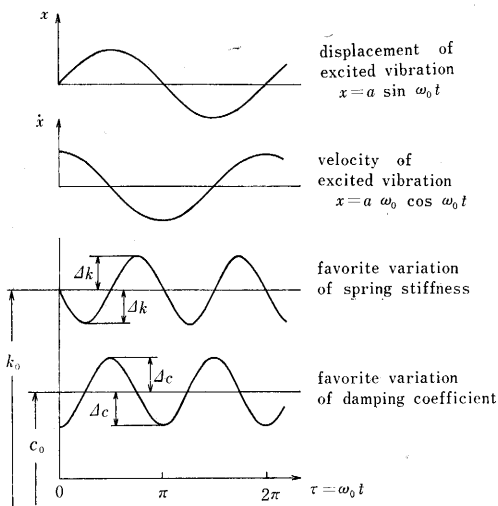


Fig. 4 Favorite variation of spring stiffness and damping coefficient with sinusoidal wave forms

variation of damping coefficient predominates over the negative effect of that of spring stiffness with the ratio of strength 4 : 1. If only the spring stiffness were variable, as some explanations misapprehend, the favorite phase relation would be opposite to that shown in Fig. 5 b, namely downward motion in central positions and upward motion in extreme positions of the swing during oscillation.

3. Double-acting and single-acting modes of excitation

The mode of excitation considered at the end of the last paragraph, namely the up-and-down motion of the centre of gravity

$$y(t) = \epsilon l_0 \sin 2\omega_0 t, \quad y(t) = L - l(t) \quad (23)$$

to excite the exponentially growing oscillation

$$\theta = a_0 e^{3\omega_0 t/4} \sin \omega_0 t \quad (24)$$

is the most effective one for the given ratio ϵ of the height variation of the centre of gravity to its distance from the sustaining axis. This mode consists of two up-and-down motions in one period of swing oscillation, and may be designated as double-acting one by analogy with the modes of acting in heat engines. Quite similarly to heat engines, the single-acting mode may also be possible and effective. It is only necessary that the periodic motion of our imaginary creature along the rod includes the 2nd harmonic with the favorite phase

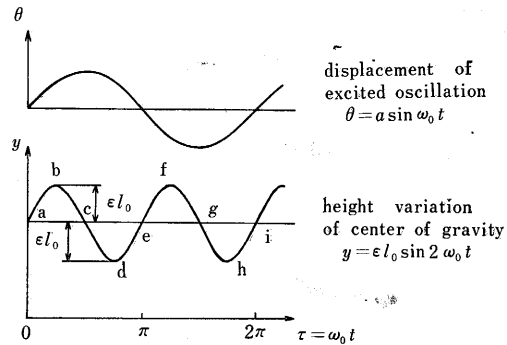


Fig. 5a Effective mode of excitation of a swing : phase relations

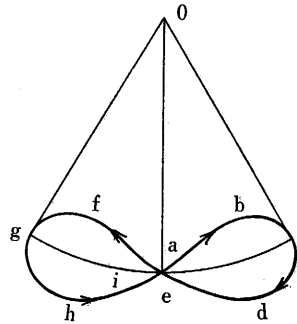


Fig. 5b Double-acting mode of excitation of swing : Locus of center of gravity relative to the swing motion.

relative to the swing motion.

One example of this mode, which is an idealized model of those widely observed in our nearby kindergartens or city parks, is illustrated in Fig. 6. It consists of an instantaneous kneeling down at either of the extreme positions and gradual standing up during the remaining whole period. The periodic functions $f(\tau)$ and $f'(\tau)$ in eq. (6) for this mode may be analysed into harmonic series as

$$\left. \begin{aligned} f(\tau) &= (1/\pi) \cos \tau - (1/2\pi) \sin 2\tau \cdots \cdots \\ f'(\tau) &= -(1/\pi) \sin \tau - (1/\pi) \cos 2\tau \cdots \cdots \end{aligned} \right\} \quad (25)$$

so that we obtain from eqs. (19) and (21)

$$a'/a = 3\epsilon/8\pi \text{ and } \theta = a_0 e^{3\omega_0 t/8\pi} \sin \omega_0 t \quad (26)$$

a_0 being the initial amplitude.

Alternatively, we may utilize only one half or one fourth of the most effective double-acting mode of excitation, as illustrated in Fig. 7a or 7b. Here we have

$$f(\tau) = (4/3\pi) \cos \tau + (1/2) \sin 2\tau + \cdots \cdots \quad (27)$$

or

$$\begin{aligned} f(\tau) &= -(1/2\pi) + (2/3\pi)(\cos \tau - \sin \tau) \\ &\quad + (1/4) \sin 2\tau + \cdots \cdots \end{aligned} \quad (28)$$

for the function $f(\tau)$ and

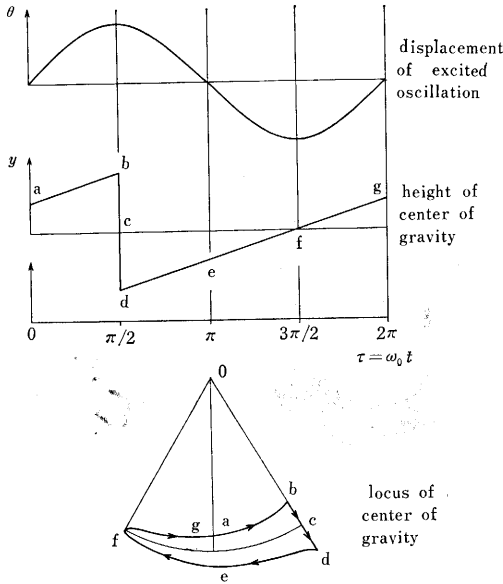


Fig. 6 Ideal model of usual mode of swing driving

$$\theta = a_0 \epsilon^{3\omega_0 t/8} \sin \omega_0 t \quad (29)$$

or

$$\theta = a_0 \epsilon^{3\omega_0 t/16} \sin \omega_0 t \quad (30)$$

for the excited oscillation.

The mode of excitations of Fig. 6 is inferior in effectiveness, but is more easily performed by a person not so accustomed to be skillful enough, than the double-acting mode. This inconveniency of the double-acting mode may be ascribed to the asymme-

try of the human body in relation to front and rear.

4. Energy balance of swing driving

The increasing sum of kinetic energy and potential energy of the steadily growing oscillation of the swing must be supplied by the driving creature sitting on it. It stands on the step and moves up and down while sustaining and resisting the component of its own weight along the rod and the centrifugal force of oscillation, which are summed up to

$$F = ml\dot{\theta}^2 + mg \cos \theta \quad (31)$$

This sum varies during one period of swing motion.

The net positive work performed by the driving creature during one period is calculated as

$$W = \oint (ml\dot{\theta}^2 + mg \cos \theta) dy \quad (32)$$

where y denotes the distance of the centre of gravity from the step and

$$\theta = a \sin \omega_0 t, \quad y = \epsilon l_0 \sin 2\omega_0 t \quad (33)$$

for our most effective double-acting mode. The result of performing this integration is given by

$$W = \epsilon m l_0^2 a^2 \omega_0^2 (\pi + (1/2)\pi) \quad (34)$$

where the one part π in the bracket is due to the change of the centrifugal force and the other part $(1/2)\pi$ comes from the change of the component of weight along the rod.

Equating this work done with the increase of the total oscillation energy during one period, namely with $(2\pi/\omega_0) d/dt ((1/2) \cdot m l_0^2 a^2 \omega_0^2)$, we have again eq. (22), namely

$$a'/a = 3\epsilon/4, \quad \theta = a_0 \epsilon^{3\omega_0 t/4} \sin \omega_0 t \quad (35)$$

as ought to be expected.

Similarly, in case of our single acting mode illustrated in Fig. 6, we have

$$\begin{aligned} W &= \oint (ml\dot{\theta}^2 + mg \cos \theta) dy \\ &= \int_{\pi/2}^{5\pi/2} (m l_0 a^2 \omega_0^2 \cos^2 \tau - (mg/2) a^2 \sin^2 \tau) (\epsilon l_0 / 2\pi) d\tau \\ &\quad + (mg/2) a^2 \epsilon l_0 \\ &= (3/4) \epsilon m l_0^2 a^2 \omega_0^2 \quad (36) \end{aligned}$$

which is equated to the total variation of the oscillation energy during

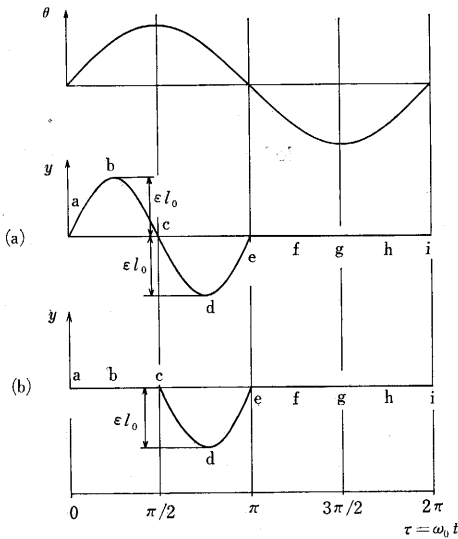


Fig. 7 Single-acting modes of excitation utilizing one half and one fourth of double-acting mode

one period to give again eq. (26), namely

$$a'/a = 3\varepsilon/8\pi, \quad \theta = a_0 e^{3\varepsilon\omega_0 t/8\pi} \sin \omega_0 t \quad (37)$$

The centrifugal force and the component of weight along the rod are both greater in the central positions than in the extreme positions. So the same stroke of up-and-down motion along the rod consumes or acquires different amount of energy according as it is performed in the central positions or in the extreme positions. This difference enables the driving creature to perform a net positive work by mere periodic up-and-down motion with a suitable phase. It reminds us of a successful stockbroker who makes a fortune out of his periodic transactions purchasing stocks in depression times and selling them in boom times of economic cycles.

5. Additional remarks

The dynamics of swing motion is usually treated as an example of applications of a Mathieu's or Hill's equation. On the contrary, our investigation is based on the energy balance only. So far as our concern is restricted to the fundamental harmonic of the excited oscillation and to the accuracy within errors of small quantity of the 1st order, our results coincide with those from exact solutions of a Mathieu's or Hill's equation. For example, the fundamental harmonic of the single-acting mode of excitation also makes some contribution to the rate of growth of the excited oscillation, but this contribution is a small quantity of the 2nd order which can be properly neglected in the presence of small original natural damping due to friction being also neglected.

The problem of a swing motion is an example of inverse problems of vibration. There exists a desired oscillation and we are requested to give effective means to realize this oscillation.

Lastly, we add some actual considerations. A swing can be driven by two persons standing face to face on the step (Fig. 8). In this case, the double-acting mode of excitation is easily performed by alternate single-acting excitation by each one. The effectiveness of this mode of excitation is equal to one half of that of double-acting mode, namely $a'/a = 3\varepsilon/8$ owing to the ratio of height

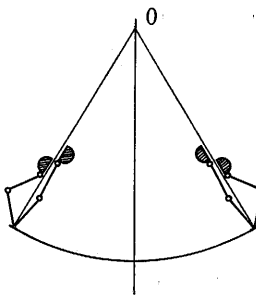


Fig. 8 Swing driven by two persons face to face

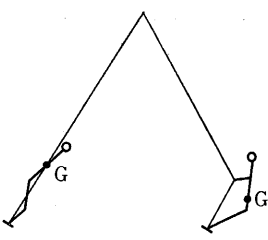


Fig. 9 Utilization of flexibility of ropes (G: Center of gravity)

variation of centre of gravity reduced to one half.

The effectiveness of swing driving depends only on the ratio ε , namely the ratio of height variation of the center of gravity to its mean distance from the sustaining axis. This circumstance probably accounts for the fact that usually a swing is composed of two flexible ropes and a step. The flexibility makes a driving person change easily the height of center of gravity (Fig. 9).

6. Conclusion

The mechanism of swing driving is reduced to the periodic change not only of the spring stiffness, but also of the damping coefficient of the oscillating system, namely a pendulum composed of a swing and a driving person. Apparent discrepancy between the theory that the favorite mode of excitation is that of the frequency twice as much as that of the pendulum and the everyday observations that usually driving persons perform their up-and-down motion only once in one period of swing oscillation, is clarified by considering that a driving mode properly called single-acting one is also possible and effective and that it is only necessary to perform the driving motion including the 2nd harmonic.

Although an exact answer to the problem of swing motion is made possible by solving a Mathieu's or Hill's equation, considerations of energy balance only are sufficient to obtain the results coincident with the exact ones as to the fundamental harmonic of the excited oscillation and within errors of small quantity of the first order where the height variation of center of gravity is regarded as a small quantity.