

Periodic Response to Sinusoidal Input of Single-loop Automatic Control System Including a Piecewise-linear Element

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Synopsis

In this paper the periodic response to a sinusoidal input of a single-loop automatic control system including a piecewise-linear element is exactly analysed by a new method, which gives the solution in the form of perfect Fourier series.

The piecewise-linear elements treated are such that their characteristics are composed of two segments of straight lines for the unsymmetrical case and of three segments of them for the symmetrical case, and the switching over conditions between these segments may have hysteresis and jump.

1. Introduction

We try to obtain the periodic response to a sinusoidal input, of a single-loop automatic control system including a piecewise-linear element, in a form of the perfect Fourier series.

The analytical methods utilized hitherto to solve the problem of steady, forced oscillation of the piecewise-linear system can be divided into three classes.

The first is the inosculating method. We obtain separately the general solutions by the usual linear processes to each of the individual linear equations corresponding to each of the segments composing the piecewise-linear characteristics, and patch up or inosculate them into one by adjusting the arbitrary constants in them to satisfy the conditions of continuity of displacement and velocity at the switching points from one segment to the other. This procedure is always possible and its result is an exact solution, without respect to that the solution can be obtained in the explicit form or not. However, the labour of patching up become very great, often up to almost prohibitive for even comparatively simple systems¹⁾, and these circumstances have hindered its general use and have given rise to the necessities of other compact if not exact methods.

The second exact method is by means of the perfect Fourier series. By this method, we dispense with the patching up procedures and can obtain the final results by one stroke. But up to recently, the application of this method have been restricted to a small number of special cases, for which the Fourier expansion of the output of the piecewise-linear element concerned is known at the outset of the problem, namely the case of the relay system²⁾, the case of the system including the spring with initial set³⁾⁴⁾, and the case of Coulomb friction⁵⁾ etc. In 1957, M.A. Aizerman gave a perfect Fourier series solution to this problem, when the characteristics of the piecewise-linear element are composed of several segments of straight lines which are all in one or other of the two given directions⁶⁾.

Meanwhile the author has been trying to analyse exactly the forced vibration of the mechanical system limited by a stop, by regarding it as that of a piecewise-linear system, and devised a new method of this type independent of the Aizerman's⁷⁾⁸⁾.

The general features of its procedure consist in :

1) Linealizing the given piecewise-linear differential equation governing the dynamic behavior of the system by expanding the nonlinear part of the output from the piecewise-linear element concerned into a Fourier series with the same

period as that of the sinusoidal input,

2) Obtaining the formal solution to thus linearized equation by regarding the above mentioned nonlinear part as if it were an input from without to the next element. This formal solution contains the unknown coefficients of the Fourier expansion above assumed,

3) Determining these unknown coefficients from the conditions that this formal solution satisfies the given piecewise-linear characteristics of the system. This reduces to solving an infinite set of simultaneous linear equations for the above mentioned coefficients as infinite number of unknowns. Some kind of convergency improvement by means of appropriate series transformation may prove advantageous in this step.

The infinity of the equations to be solved to determine the unknown Fourier coefficients is a weak point of the author's method as compared to the Alzerman's, where the determination of them is reduced to solving the simultaneous equations of $k \times n$ in number, k being the number of segments composing the piecewise-linear characteristics and n being the order of the differential equation governing the dynamic behavior of the system.

However, the unique existence of the solution and the convergence to the exact value of the approximate solution by curtailment of the higher orders of unknowns and equations are both assured with exceptional cases. And, by appropriate series transformation, the degree of convergency can be improved greatly such that for practical accuracy the number of equations to be solved may be reduced to only small, three or so.

And the coefficients and constants of the infinite set of equations are given in finite terms as the functions of the frequency of the input and the nonlinear ratio (the ratio of the nonlinear interval to the whole period), whose functional forms are the same for the same type of piecewise-linearity and independent of the degree of nonlinearity. So, if they are once calculated as the functions of both parameters, they may be utilized for all cases of the same type of piecewise-linearity.

The applicability is not confined to the case of only one piecewise-linear element in the system. In principle, our method is applicable to the case of any number of them in the system, but the labour increases rapidly with the number of switching points.

As the third, we consider the miscellaneous approximative methods, which include various two-terms approximation methods by Fourier series such as the describing function method, various graphical methods such as the phase plane method, and various analog methods such as that by means of the analog computer.

Any one of these approximative methods is very powerful and convenient for the particular problem concerned, but because of its approximate or analog character it lacks the general outlook and often leads to quantitative or even qualitative errors.

2. Block Diagram and Formal Solution

The block diagram of the system to be treated in this paper is shown in Fig. 1. In it, the only

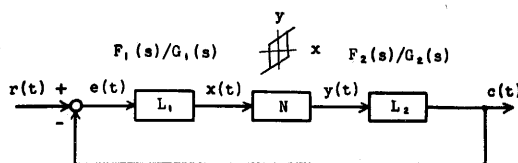


Fig. 1

nonlinear element N is of piecewise-linear characteristic $y=f(x)$, which is shown in Fig. 2a or Fig. 2b according as it is unsymmetrical or symmetrical. Namely, the whole phase space representing the dynamic states of the system is divided into two regions (I) and (II) for the unsymmetrical case, or into three regions (I), (II) and

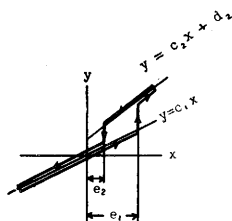


Fig. 2a

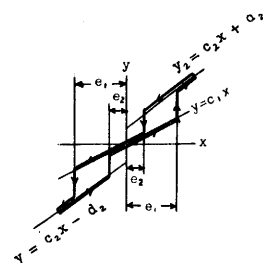


Fig. 2b

(III), symmetrical to (II), for the symmetrical case, that is

$$\left. \begin{aligned} f(x) &= c_1 x & , & \text{Region (I)} \\ f(x) &= c_2 x + d_2 & , & \text{Region (II)} \\ f(x) &= c_2 x - d_2 & , & \text{Region (III)} \end{aligned} \right\} \quad (1)$$

where the letters in the parentheses () refer only to the symmetrical case, which provision is to be obeyed throughout in this paper.

The conditions of switching over between two or three regions are accompanied by hysteresis and jump as shown in Fig. 2a and Fig. 2b and can be written as

$$\left. \begin{aligned} x = e_1, x > 0, & \text{Region (I)} \rightarrow \text{Region (II)} \\ x = e_2, x < 0, & \text{Region (II)} \rightarrow \text{Region (I)} \\ (x = -e_1, x < 0, & \text{Region (I)} \rightarrow \text{Region (III)}) \\ (x = -e_2, x > 0, & \text{Region (III)} \rightarrow \text{Region (I)}) \end{aligned} \right\} \quad (2)$$

The linear elements lying before and after N are compressed into two equivalent linear elements L_1 and L_2 , whose transfer functions are put equal to $F_1(s)/G_1(s)$ and $F_2(s)/G_2(s)$ respectively, where $F_1(s)$, $F_2(s)$, $G_1(s)$ and $G_2(s)$ are all certain polynomials in s , and the degree of $G_2(s)$ is higher than that of $F_2(s)$ and that of $G_2(s) \times G_1(s)$ is higher than that of $F_2(s) \times F_1(s)$.

And, $r(t)$ and $c(t)$ are the input and output of the whole system, $e(t)$ is the error signal, and $x(t)$ and $y(t)$ are the input and the output of the piecewise-linear element concerned.

According to this block diagram, we can write down the differential equation governing the dynamic behavior of the system as

$$\frac{G_2(p)G_1(p)}{F_2(p)F_1(p)} x(t) + f\{x(t)\} = \frac{G_2(p)}{F_2(p)} r(t) \quad (3)$$

where p means the differential operator d/dt with respect to time t .

In this paper we confine our analysis to the case of the sinusoidal input,

$$r(t) = f \cos \omega t, \quad (4)$$

where f and ω are its amplitude and circular frequency.

Then, if we define B and δ by

$$Be^{-j\delta} = \frac{G_2(j\omega)}{F_2(j\omega)} f, \quad j = \sqrt{-1}, \quad (5)$$

the equation (3) reduces to

$$\frac{G_2(p)G_1(p)}{F_2(p)F_1(p)} x + f(x) = B \cos(\omega t - \delta) \quad (6)$$

Written separately for each of regions, this is

equivalent to following two (three) equations,

$$\left. \begin{aligned} \frac{G_2(p)G_1(p)}{F_2(p)F_1(p)} x + c_1 x &= B \cos(\omega t - \delta), & \text{Region (I)} \\ \frac{G_2(p)G_1(p)}{F_2(p)F_1(p)} x + c_2 x + d_2 &= B \cos(\omega t - \delta), & \text{Region (II)} \\ \frac{G_2(p)G_1(p)}{F_2(p)F_1(p)} x + c_2 x - d_2 &= B \cos(\omega t - \delta), & \text{Region (III)} \end{aligned} \right\} \quad (7)$$

After some transients, the response becomes steady and periodic, and then the whole period may be divided into several intervals, during each of which the system stays in one of two (three) regions.

We consider only the periodic response of the period equal to that of the input, and further, we confine ourselves to the case of two (four) intervals, namely of two (four) switching points only. When the subharmonic resonance occurs, the period of the response may be several times as long as that of the input, and in the case of the superharmonic resonance, for example, the number of intervals may be much greater than that of the regions, that is the system may join each region more than once in one period.

We regard the interval (I) as standard, then the interval (II) (and (III)) is considered as nonlinear, and we put the nonlinear part of the output $y(t)$ of the nonlinear element N equal to $g(\theta)$, θ being the phase angle measured from the middle point of the interval (II). This phase angle is behind that of the input by the phase lag angle α , which is to be determined afterwards,

$$\theta = \omega t - \alpha. \quad (8)$$

And if we put the length of the interval (II) (and so of the interval (III)), equal to θ_0 as

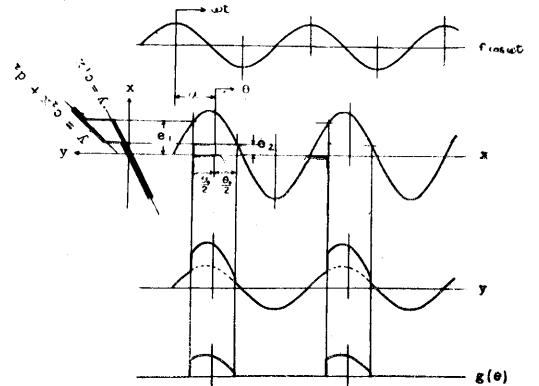


Fig. 3

shown in Fig. 3, the conditions for the piecewise-linear characteristics become

$$\left. \begin{aligned} g(\theta) &= (c_2 - c_1)x + d_2 = K \left(x - \frac{e_1 + e_2}{2} \right) + H, \quad \frac{\theta_0}{2} > \theta > -\frac{\theta_0}{2} \\ g(\theta) &= 0, \quad [2\pi - \frac{\theta_0}{2} > \theta > \frac{\theta_0}{2}, \quad (2\pi - \frac{\theta_0}{2} > \theta > \pi + \frac{\theta_0}{2}) \\ g(\theta) &= (c_2 - c_1)x - d_2 = K \left(x + \frac{e_1 + e_2}{2} \right) - H, \quad \pi + \frac{\theta_0}{2} > \theta > \pi - \frac{\theta_0}{2} \end{aligned} \right\} \quad (9)$$

where K and H are defined by

$$K \equiv c_2 - c_1, \quad H \equiv d_2 + (c_2 - c_1) \frac{e_1 + e_2}{2} \quad (10)$$

and the figures in the square brackets [] refer only to the unsymmetrical case, which provision too is to be obeyed throughout in this paper.

The ratio $\theta_0/2\pi$ (θ_0/π) of the nonlinear interval (of the sum of the nonlinear intervals) to the whole period is designated as the nonlinear ratio and appears to be an important parameter in the following analysis.

By utilizing $g(\theta)$ and by changing the independent variable from t to θ , we can combine two (three) equations of (7) into one, that is,

$$\frac{G_2(\omega q)G_1(\omega q)}{F_2(\omega q)F_1(\omega q)}x + c_1x = B \cos(\theta + \beta) - g(\theta), \quad (11)$$

where q represents the differential operator $d/d\theta$ with respect to θ and

$$\beta = \alpha - \delta. \quad (12)$$

As the nonlinear part $g(\theta)$ of the output is the periodic function of period 2π , it can be expanded into a Fourier series, so that

$$g(\theta) = [a_0] + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad (13)$$

where $\sum_{n=1}^{\infty}$ means summation over all positive integers $\sum_{n=1,2,3,\dots}$ for the unsymmetrical case or summation over all positive odd integers $\sum_{n=1,3,5,\dots}$ for the symmetrical case, and the square bracket [] means, as mentioned before, that it stands only for the unsymmetrical case.

According to the equation (11), we consider the equivalent linear system as shown in Fig. 4,

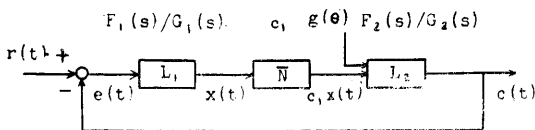


Fig. 4

in which the nonlinear element N is replaced by the linear one \bar{N} of characteristic $y = c_1x$ and $g(\theta)$ is regarded as an input from without to the next element L_2 .

The expansion (13) being introduced into (11), the original nonlinear equation becomes, so to speak, linearized formally and transformed into

$$\frac{G_2(\omega q)G_1(\omega q)}{F_2(\omega q)F_1(\omega q)}x + c_1x = B \cos(\theta + \beta) - [a_0] - \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad (14)$$

If we define M_n , φ_n , R_n and X_n ($n=0,1,2,3,\dots$) by

$$M_n e^{-j\varphi_n} \equiv \frac{1}{R_n + jX_n} \equiv \frac{k}{\frac{G_2(jn\omega)G_1(jn\omega)}{F_2(jn\omega)F_1(jn\omega)} + c_1}, \quad (15)$$

where $k = c_1$, when $c_1 \neq 0$,

or $k = K$, when $c_1 = 0$ and $c_2 \neq 0$,

or $k = \text{appropriate constant of the same dimension to the denominator of (15), when } c_1 = 0 \text{ and } c_2 = 0$, we can write down the formal solution of (11) in the following form,

$$x = M_1 \frac{B}{k} \cos(\theta + \beta - \varphi_1) - M_0 \frac{[a_0]}{k} - \sum_{n=1}^{\infty} M_n \left\{ \frac{a_n}{k} \cos(n\theta - \varphi_n) + \frac{b_n}{k} \sin(n\theta - \varphi_n) \right\} \quad (16)$$

Meanwhile, the conditions of switching over become

$$\left. \begin{aligned} \theta &= -\frac{\theta_0}{2}, \quad x = e_1, \quad \dot{x} > 0, \quad (\text{I}) \rightarrow (\text{II}) \\ \theta &= \frac{\theta_0}{2}, \quad x = e_2, \quad \dot{x} < 0, \quad (\text{II}) \rightarrow (\text{I}) \\ \theta &= \pi - \frac{\theta_0}{2}, \quad x = -e_1, \quad \dot{x} < 0, \quad (\text{I}) \rightarrow (\text{III}) \\ \theta &= \pi + \frac{\theta_0}{2}, \quad x = -e_2, \quad \dot{x} > 0, \quad (\text{III}) \rightarrow (\text{I}) \end{aligned} \right\} \quad (17)$$

and these are all satisfied by the following two equations

$$\left. \begin{aligned} e_1 \\ e_2 \end{aligned} \right\} = M_1 \frac{B}{k} \cos(\beta - \varphi_1) \cos \frac{\theta_0}{2} \pm M_1 \frac{B}{k} \sin(\beta - \varphi_1) \sin \frac{\theta_0}{2} - M_0 \frac{[a_0]}{k} - \sum_{n=1}^{\infty} M_n \left(\frac{a_n}{k} \cos \varphi_n - \frac{b_n}{k} \sin \varphi_n \right) \cos \frac{n\theta_0}{2} \pm \sum_{n=1}^{\infty} M_n \left(\frac{a_n}{k} \sin \varphi_n + \frac{b_n}{k} \cos \varphi_n \right) \sin \frac{n\theta_0}{2} \quad (18)$$

From these, we obtain

$$\left. \begin{aligned} \frac{e_1 + e_2}{2} &= M_1 \frac{B}{k} \cos(\beta - \varphi_1) \cos \frac{\theta_0}{2} - M_0 \frac{[a_0]}{k} - \sum_{n=1}^{\infty} M_n \left(\frac{a_n}{k} \cos \varphi_n - \frac{b_n}{k} \sin \varphi_n \right) \cos \frac{n\theta_0}{2} \\ \frac{e_2 - e_1}{2} &= -M_1 \frac{B}{k} \sin(\beta - \varphi_1) \sin \frac{\theta_0}{2} - \sum_{n=1}^{\infty} M_n \left(\frac{a_n}{k} \sin \varphi_n + \frac{b_n}{k} \cos \varphi_n \right) \sin \frac{n\theta_0}{2} \end{aligned} \right\} \quad (19)$$

And further, these introduced into (16), the expression for $x(t)$ becomes

$$\left. \begin{aligned} x = & \left\{ \frac{e_1 + e_2}{2} + M_0 \frac{[a_0]}{k} + \sum_{n=2}^{\infty} M_n \left(\frac{a_n}{k} \cos \varphi_n - \frac{b_n}{k} \sin \varphi_n \right) \cos \frac{n\theta_0}{2} \right\} \frac{\cos \theta}{\cos \frac{\theta_0}{2}} \\ & + \left\{ \frac{e_2 - e_1}{2} + \sum_{n=2}^{\infty} M_n \left(\frac{a_n}{k} \sin \varphi_n + \frac{b_n}{k} \cos \varphi_n \right) \sin \frac{n\theta_0}{2} \right\} \frac{\sin \theta}{\sin \frac{\theta_0}{2}} \\ & - M_0 \frac{[a_0]}{k} - \sum_{n=2}^{\infty} M_n \left(\frac{a_n}{k} \cos \varphi_n - \frac{b_n}{k} \sin \varphi_n \right) \cos n\theta \\ & - \sum_{n=2}^{\infty} M_n \left(\frac{a_n}{k} \sin \varphi_n + \frac{b_n}{k} \cos \varphi_n \right) \sin n\theta \end{aligned} \right\} \quad (20)$$

The fundamental harmonic in this expression can be written as,

$$A \cos \gamma \cos \theta + A \sin \gamma \sin \theta = A \cos(\theta - \gamma), \quad (21)$$

where

$$\left. \begin{aligned} A \cos \gamma &= \left\{ \frac{e_1 + e_2}{2} + M_0 \frac{[a_0]}{k} + \sum_{n=2}^{\infty} M_n \left(\frac{a_n}{k} \cos \varphi_n - \frac{b_n}{k} \sin \varphi_n \right) \cos \frac{n\theta_0}{2} \right\} \frac{1}{\cos \frac{\theta_0}{2}} \\ A \sin \gamma &= \left\{ \frac{e_2 - e_1}{2} + \sum_{n=2}^{\infty} M_n \left(\frac{a_n}{k} \sin \varphi_n + \frac{b_n}{k} \cos \varphi_n \right) \sin \frac{n\theta_0}{2} \right\} \frac{1}{\sin \frac{\theta_0}{2}} \end{aligned} \right\} \quad (22)$$

And, these notations being used, the two equations (19) are transformed into

$$\left. \begin{aligned} A \cos \gamma &= M_1 \frac{B}{k} \cos(\beta - \varphi_1) - M_1 \left(\frac{a_1}{k} \cos \varphi_1 - \frac{b_1}{k} \sin \varphi_1 \right) \\ A \sin \gamma &= -M_1 \frac{B}{k} \sin(\beta - \varphi_1) - M_1 \left(\frac{a_1}{k} \sin \varphi_1 + \frac{b_1}{k} \cos \varphi_1 \right) \end{aligned} \right\} \quad (23)$$

If we nondimensionalize the assumed Fourier coefficients a_0 , a_n and b_n by

$$x_0 \equiv a_0/kA, \quad x_n \equiv a_n/kA, \quad y_n \equiv b_n/kA \quad (24)$$

namely, by

$$g(\theta)/kA = [x_0] + \sum_{n=1}^{\infty} (x_n \cos n\theta + y_n \sin n\theta), \quad (25)$$

the expression for $x(t)$ becomes

$$x(t) = A \left(\begin{aligned} & \cos \gamma \cos \theta + \sin \gamma \sin \theta - M_0 [x_0] \\ & - \sum_{n=2}^{\infty} M_n (x_n \cos \varphi_n - y_n \sin \varphi_n) \cos n\theta \\ & - \sum_{n=2}^{\infty} M_n (x_n \sin \varphi_n + y_n \cos \varphi_n) \sin n\theta \end{aligned} \right) \quad (26)$$

and the formulas (22) defining A and γ become

$$\left. \begin{aligned} \cos \frac{\theta_0}{2} \cos \gamma &= \frac{e_1 + e_2}{2A} + M_0[x_0] + \sum_{n=2}^{\infty} M_n(x_n \cos \varphi_n - y_n \sin \varphi_n) \cos \frac{n\theta_0}{2} \\ \sin \frac{\theta_0}{2} \sin \gamma &= \frac{e_2 - e_1}{2A} + \sum_{n=2}^{\infty} M_n(x_n \sin \varphi_n + y_n \cos \varphi_n) \sin \frac{n\theta_0}{2}, \end{aligned} \right\} \quad (27)$$

and lastly the conditions (23) deduced from the conditions of switching over are written as

$$\left. \begin{aligned} A\{\cos \gamma + M_1(x_1 \cos \varphi_1 - y_1 \sin \varphi_1)\} &= M_1 \frac{B}{k} \cos(\beta - \varphi_1) \\ A\{\sin \gamma + M_1(x_1 \sin \varphi_1 + y_1 \cos \varphi_1)\} &= -M_1 \frac{B}{k} \sin(\beta - \varphi_1) \end{aligned} \right\} \quad (28)$$

which can be further transformed into

$$\left. \begin{aligned} R_1 + x_1 \cos \gamma + y_1 \sin \gamma &= \frac{B}{kA} \cos(\beta + \gamma) \\ X_1 + x_1 \sin \gamma - y_1 \cos \gamma &= \frac{B}{kA} \sin(\beta + \gamma) \end{aligned} \right\} \quad (29)$$

Thus, we obtain, as the final expressions for the amplitude A and the phase lag angle $\alpha + \gamma$, the following two equations,

$$A = \frac{B}{k} \frac{1}{\sqrt{(R_1 + x_1 \cos \gamma + y_1 \sin \gamma)^2 + (X_1 + x_1 \sin \gamma - y_1 \cos \gamma)^2}} \quad (30)$$

$$\alpha + \gamma = \delta + \tan^{-1} \frac{X_1 + x_1 \sin \gamma - y_1 \cos \gamma}{R_1 + x_1 \cos \gamma + y_1 \sin \gamma} \quad (31)$$

If we discard higher harmonics and leave the fundamental harmonic only, we obtain, corresponding to the input to the nonlinear element N ,

$$x(t) = A \cos(\theta - \gamma) \quad (32)$$

the output

$$\left. \begin{aligned} y(t) &= c_1 x(t) + [a_0] + a_1 \cos \theta + b_1 \sin \theta, \\ &= [a_0] + P_1 \cos(\theta - \gamma) - Q_1 \sin(\theta - \gamma), \end{aligned} \right\} \quad (33)$$

where

$$\left. \begin{aligned} P_1 &= c_1 A + a_1 \cos \gamma + b_1 \sin \gamma \\ Q_1 &= a_1 \sin \gamma - b_1 \cos \gamma \end{aligned} \right\} \quad (34)$$

The describing function is defined for this case as

$$\left. \begin{aligned} \frac{P_1 + jQ_1}{A} &= c_1 + k(x_1 - jy_1)e^{j\gamma} \\ &= c_1 + k(x_1 \cos \gamma + y_1 \sin \gamma) \\ &\quad + jk(x_1 \sin \gamma - y_1 \cos \gamma) \end{aligned} \right\} \quad (35)$$

Utilizing (25), (26) and (27), we can write the conditions of piecewise-linear characteristics (9) in nondimensional form as follows,

$$\left. \begin{aligned} &[x_0] + \sum_{n=1}^{\infty} (x_n \cos n\theta + y_n \sin n\theta) \\ &= \frac{K}{k} \left\{ \cos \gamma \left(\cos \theta - \cos \frac{\theta_0}{2} \right) + \sin \gamma \sin \theta \right. \\ &\quad \left. - \sum_{n=2}^{\infty} M_n(x_n \cos \varphi_n - y_n \sin \varphi_n) \left(\cos n\theta - \cos \frac{n\theta_0}{2} \right) \right. \\ &\quad \left. - \sum_{n=2}^{\infty} M_n(x_n \sin \varphi_n + y_n \cos \varphi_n) \sin n\theta \right\} + \frac{H}{kA} \\ &\quad \left. \frac{\theta_0}{2} > \theta > -\frac{\theta_0}{2} \right\} \\ &[x_0] + \sum_{n=1}^{\infty} (x_n \cos n\theta + y_n \sin n\theta) = 0, \quad [2\pi - \frac{\theta_0}{2} > \theta > \frac{\theta_0}{2}] \end{aligned} \right\} \quad (36)$$

The second equation for the interval $(2\pi - \frac{\theta_0}{2} > \theta > \pi + \frac{\theta_0}{2})$ and the last equation in (9) is automatically satisfied by the symmetric character of the Fourier series for $g(\theta)$, that is,

$$(g(\theta + \pi) = -g(\theta)) \quad (37)$$

As explained in the following paragraph, the procedures similar to those obtaining the Fourier coefficients for an arbitrary function, determine the nondimensional coefficients x_0 , x_n and y_n in the following forms,

$$\begin{pmatrix} x_0 \\ x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_0' \\ x_n' \\ y_n' \end{pmatrix} \cos \gamma + \begin{pmatrix} x_0'' \\ x_n'' \\ y_n'' \end{pmatrix} \sin \gamma + \begin{pmatrix} x_0''' \\ x_n''' \\ y_n''' \end{pmatrix} \frac{H}{kA}, \quad (38)$$

where, for the given system, three infinite set of nondimensional coefficients (x_0', x_n', y_n') , (x_0'', x_n'', y_n'') and (x_0''', x_n''', y_n''') are all functions of the input frequency ω and the nonlinear ratio $\theta_0/[2]\pi$ only.

If we use next notations for the sake of simplicity,

$$\left. \begin{aligned} S' &= \sum_{n=2}^{\infty} M_n(x_n' \cos \varphi_n - y_n' \sin \varphi_n) \cos \frac{n\theta_0}{2} + M_0[x_0] \\ T' &= \sum_{n=2}^{\infty} M_n(x_n' \sin \varphi_n + y_n' \cos \varphi_n) \sin \frac{n\theta_0}{2} \end{aligned} \right\} \quad (39)$$

and further two similar ones S'' , T'' and S''' , T''' , the equations (27) become transformed into

$$\left. \begin{aligned} (\cos \frac{\theta_0}{2} - S') A \cos \gamma - S'' A \sin \gamma &= \frac{e_1 + e_2}{2} + S''' \frac{H}{k} \\ -T' A \cos \gamma + (\sin \frac{\theta_0}{2} - T'') A \sin \gamma &= \frac{e_2 - e_1}{2} + T''' \frac{H}{h} \end{aligned} \right\} \quad (40)$$

Adding after squaring and dividing one by the other of the solutions of these equations for $A \cos \gamma$ and $A \sin \gamma$ as two unknowns give

$$A = \sqrt{\left| \begin{array}{cc} \frac{e_1 + e_2}{2} + S''' \frac{H}{k}, & -S'' \\ \frac{e_2 - e_1}{2} + T''' \frac{H}{k}, & \sin \frac{\theta_0}{2} - T'' \end{array} \right|^2 + \left| \begin{array}{cc} \cos \frac{\theta_0}{2} - S', & \frac{e_1 + e_2}{2} + S''' \frac{H}{k} \\ -T', & \frac{e_2 - e_1}{2} + T''' \frac{H}{k} \end{array} \right|^2} \quad (41)$$

and

$$\gamma = \tan^{-1} \frac{\left| \begin{array}{cc} \cos \frac{\theta_0}{2} - S', & \frac{e_1 + e_2}{2} + S''' \frac{H}{k} \\ -T', & \frac{e_2 - e_1}{2} + T''' \frac{H}{k} \end{array} \right|}{\left| \begin{array}{cc} \frac{e_1 + e_2}{2} + S''' \frac{H}{k}, & -S'' \\ \frac{e_2 - e_1}{2} + T''' \frac{H}{k}, & \sin \frac{\theta_0}{2} - T'' \end{array} \right|} \quad (42)$$

For the given system, H , e_1 and e_2 being given, the right sides of the equations (41) and (42) are given as the functions of the input frequency and the nonlinear ratio, and according to these the input amplitude f is also given as the function of these two parameters by (30), that is

$$f = k \left| \frac{F_2(j\omega)}{G_2(j\omega)} \right| A \sqrt{(R_1 + x_1 \cos \gamma + y_1 \sin \gamma)^2 + (X_1 + x_1 \sin \gamma - y_1 \cos \gamma)^2} \quad (43)$$

where the expressions for various coefficients are

$$\left. \begin{aligned} \gamma_{00}=0, \quad \gamma_{01}=0, \quad \gamma_{0n} &= -\frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{n\theta_0}{2}}{\frac{n\theta_0}{2}} - \cos \frac{n\theta_0}{2} \right\} M_n \cos \varphi_n, \quad (n \geq 2) \\ \gamma_{m0}=0, \quad \gamma_{m1}=0, \quad \gamma_{mn} &= -\frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{m+n}{2} \theta_0}{\frac{m+n}{2} \theta_0} + \frac{\sin \frac{m-n}{2} \theta_0}{\frac{m-n}{2} \theta_0} \right. \\ &\quad \left. - 2 \frac{\sin \frac{m\theta_0}{2}}{\frac{m\theta_0}{2}} \cos \frac{n\theta_0}{2} \right\} M_n \cos \varphi_n, \quad (n \geq 2) \end{aligned} \right\} \quad (46)$$

$$\left. \begin{aligned} \delta_{01}=0, \quad \delta_{0n} &= \frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{n\theta_0}{2}}{\frac{n\theta_0}{2}} - \cos \frac{n\theta_0}{2} \right\} M_n \sin \varphi_n, \quad (n \geq 2) \\ \delta_{m1}=0, \quad \delta_{mn} &= \frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{m+n}{2} \theta_0}{\frac{m+n}{2} \theta_0} + \frac{\sin \frac{m-n}{2} \theta_0}{\frac{m-n}{2} \theta_0} \right. \\ &\quad \left. - 2 \frac{\sin \frac{m\theta_0}{2}}{\frac{m\theta_0}{2}} \cos \frac{n\theta_0}{2} \right\} M_n \sin \varphi_n, \quad (n \geq 2) \end{aligned} \right\} \quad (47)$$

$$\varepsilon_{m0}=0, \quad \varepsilon_{m1}=0, \quad \varepsilon_{mn} = -\frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{m-n}{2} \theta_0}{\frac{m-n}{2} \theta_0} - \frac{\sin \frac{m+n}{2} \theta_0}{\frac{m+n}{2} \theta_0} \right\} M_n \sin \varphi_n, \quad (n \geq 2) \quad (48)$$

$$\zeta_{m1}=0, \quad \zeta_{mn} = -\frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{m-n}{2} \theta_0}{\frac{m-n}{2} \theta_0} - \frac{\sin \frac{m+n}{2} \theta_0}{\frac{m+n}{2} \theta_0} \right\} M_n \cos \varphi_n, \quad (n \geq 2) \quad (49)$$

($m=1, [2], 3, \dots$)

And the constants in the right sides can be written as

$$\begin{pmatrix} \alpha_0 \\ \alpha_m \\ \beta_m \end{pmatrix} = \frac{K}{k} \begin{pmatrix} \alpha_0' \\ \alpha_m' \\ \beta_m' \end{pmatrix} \cos \gamma + \frac{K}{k} \begin{pmatrix} \alpha_0'' \\ \alpha_m'' \\ \beta_m'' \end{pmatrix} \sin \gamma + \begin{pmatrix} \alpha_0''' \\ \alpha_m''' \\ \beta_m''' \end{pmatrix} \frac{H}{kA} \quad (50)$$

where

$$\begin{pmatrix} \alpha_0' \\ \alpha_m' \\ \beta_m' \end{pmatrix} = \begin{pmatrix} \frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{\theta_0}{2}}{\frac{\theta_0}{2}} - \cos \frac{\theta_0}{2} \right\} \\ \frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{m+1}{2} \theta_0}{\frac{m+1}{2} \theta_0} + \frac{\sin \frac{m-1}{2} \theta_0}{\frac{m-1}{2} \theta_0} - 2 \frac{\sin \frac{m\theta_0}{2}}{\frac{m\theta_0}{2}} \cos \frac{\theta_0}{2} \right\} \\ 0 \end{pmatrix} \quad (51)$$

$$\begin{pmatrix} \alpha_0'' \\ \alpha_m'' \\ \beta_m'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\theta_0}{[2]\pi} \left\{ \frac{\sin \frac{m-1}{2} \theta_0}{\frac{m-1}{2} \theta_0} - \frac{\sin \frac{m+1}{2} \theta_0}{\frac{m+1}{2} \theta_0} \right\} \end{pmatrix} \quad (52)$$

$$\begin{pmatrix} \alpha_0''' \\ \alpha_m''' \\ \beta_m''' \end{pmatrix} = \begin{pmatrix} \frac{\theta_0}{[2]\pi} \\ \frac{2\theta_0}{[2]\pi} \frac{\sin \frac{m\theta_0}{2}}{\frac{m\theta_0}{2}} \\ 0 \end{pmatrix} \quad (53)$$

Owing to the linear property of the set of equations (45), its solution can be written into similar forms to the constants in the right sides, that is,

$$\begin{pmatrix} x_0 \\ x_m \\ y_m \end{pmatrix} = \begin{pmatrix} x_0' \\ x_m' \\ y_m' \end{pmatrix} \cos \gamma + \begin{pmatrix} x_0'' \\ x_m'' \\ y_m'' \end{pmatrix} \sin \gamma + \begin{pmatrix} x_0''' \\ x_m''' \\ y_m''' \end{pmatrix} \frac{H}{kA} \quad (54)$$

where (x_0', x_m', y_m') , (x_0'', x_m'', y_m'') and (x_0''', x_m''', y_m''') are solutions of (45), when $(\frac{K}{k}\alpha_0', \frac{K}{k}\alpha_m', \frac{K}{k}\beta_m')$, $(\frac{K}{k}\alpha_0'', \frac{K}{k}\alpha_m'', \frac{K}{k}\beta_m'')$ and $(\alpha_0''', \alpha_m''', \beta_m''')$ are substituted for $(\alpha_0, \alpha_m, \beta_m)$ respectively.

The existence of the solution for the infinite set of simultaneous linear equations (45) and the convergence to the exact value of the approximative solutions by discarding equations and unknowns of higher orders are both assured with rare exceptional cases. These problems are briefly discussed in the appendix.

Due to the poor convergency of the sequence x_0, x_n and y_n , the accurate determination of them from the equations (45) is very laborious if not impossible, and the direct method is more suitable for the calculation of the general character such as the amplitude and the phase lag angle of the fundamental harmonic, than the determination of the minute structure, such as the wave forms.

As the first approximation for the direct method, we discard the higher harmonics thoroughly, namely we neglect x_n and y_n for $n \geq 2$, then the solution takes very simple form, so that

$$\left\{ \begin{aligned} x_0 &= \frac{K}{k} \frac{\theta_0}{2\pi} \left[\frac{\sin \frac{\theta_0}{2}}{\frac{\theta_0}{2}} - \cos \frac{\theta_0}{2} \right] \cos \gamma + \frac{\theta}{2\pi} \frac{H}{kA} \\ x_1 &= \frac{K}{k} \frac{\theta_0}{[2]\pi} \left(1 - \frac{\sin \theta_0}{\theta_0} \right) \cos \gamma + \frac{2\theta_0}{[2]\pi} \frac{\sin \frac{\theta_0}{2}}{\frac{\theta_0}{2}} \frac{H}{kA} \\ y_1 &= \frac{K}{k} \frac{\theta_0}{[2]\pi} \left(1 - \frac{\sin \theta_0}{\theta_0} \right) \sin \gamma \end{aligned} \right\} \quad (55)$$

And, as $S', S'', S''', T', T'',$ and T''' all reduce to zero in this case, (41) and (42) become

$$A = \sqrt{\left(\frac{e_1 + e_2}{2} \frac{1}{\cos \frac{\theta_0}{2} - M_0[x_0]} \right)^2 + \left(\frac{e_2 - e_1}{2 \sin \frac{\theta_0}{2}} \right)^2} \quad (56)$$

$$\tan \gamma = \frac{e_2 - e_1}{e_2 + e_1} \frac{\cos \frac{\theta_0}{2} - M_0[x_0]}{\sin \frac{\theta_0}{2}} \quad (57)$$

These with (43) determine the nonlinear ratio and complete the solution.

4. Determination of Nondimensional Coefficients — Convergency Improvement by Series Transformation

As explained in the last paragraph, the poor convergency of the sequence x_0, x_n and y_n hinders

the full knowledge of the response. So, we are to improve the convergency by transforming the original series (25) into the new one, for the interval $(-\theta_0/2, \theta_0/2)$, which is

$$g(\theta)/kA = \xi_0 + \sum_{m=1,3,5,\dots} (\xi_m \cos \frac{m\pi}{\theta_0} \theta + \gamma_m \sin \frac{m\pi}{\theta_0} \theta) \quad (58)$$

The relation between the coefficients of the new and old series can be obtained from the equality of the old series (25) and the new series (58) in the interval $(-\theta_0/2, \theta_0/2)$, so that

$$\left. \begin{aligned} x_0 &= \frac{\theta_0}{2\pi} \left\{ \xi_0 + \sum_{m=1,3,5,\dots} \frac{(-1)^{\frac{m-1}{2}}}{m\pi} \xi_m \right\} \\ x_n &= \frac{4}{[2]n\pi} \left\{ \xi_0 \sin \frac{n\theta_0}{2} + \cos \frac{n\theta_0}{2} \sum_{m=1,3,5,\dots} \frac{(-1)^{\frac{m-1}{2}}}{\frac{m^2\pi^2}{\theta_0^2} - n^2} \xi_m \right\} \\ y_n &= \frac{4}{[2]\pi} \cos \frac{n\theta_0}{2} \sum_{m=1,3,5,\dots} \frac{(-1)^{\frac{m-1}{2}}}{\frac{m^2\pi^2}{\theta_0^2} - n^2} \gamma_m \end{aligned} \right\} \quad (59)$$

And $\cos n\theta$ and $\sin n\theta$ can also be expanded into Fourier series in the above interval as

$$\left. \begin{aligned} \cos n\theta &= \cos \frac{n\theta_0}{2} + \frac{\cos \frac{n\theta_0}{2}}{\frac{n\theta_0}{2}} \sum_{l=1,3,5,\dots} \frac{(-1)^{\frac{l-1}{2}} 2n^3}{l\pi \left\{ \frac{l^2\pi^2}{\theta_0^2} - n^2 \right\}} \cos \frac{l\pi}{\theta_0} \theta \\ \sin n\theta &= \frac{\cos \frac{n\theta_0}{2}}{\frac{n\theta_0}{2}} \sum_{l=1,3,5,\dots} \frac{(-1)^{\frac{l-1}{2}} 2n^2}{\frac{l^2\pi^2}{\theta_0^2} - n^2} \sin \frac{l\pi}{\theta_0} \theta \end{aligned} \right\} \quad (60)$$

When we substitute these expressions of (59) and (60) for those in the first equation of (36), the left side becomes equal to (58) and the right side becomes also the Fourier series of $\cos m \frac{\pi}{\theta_0} \theta$ and $\sin m \frac{\pi}{\theta_0} \theta$. Because these two sides are both valid for the interval $(-\theta_0/2, \theta_0/2)$, they are two different expressions for the same Fourier series. Equating the coefficients of $\cos m \frac{\pi}{\theta_0} \theta$ and $\sin m \frac{\pi}{\theta_0} \theta$ in both sides, we obtain again an infinite set of simultaneous linear equations for ξ_m, γ_m as unknowns,

$$\left. \begin{aligned} (1 - \frac{K}{k} \gamma_{00}) \xi_0 - \frac{K}{k} \gamma_{01} \xi_1 - \frac{K}{k} \gamma_{03} \xi_3 - \dots & \dots - \frac{K}{k} \delta_{01} \gamma_1 - \frac{K}{k} \delta_{03} \gamma_3 - \dots = \alpha_0 \\ - \frac{K}{k} \gamma_{10} \xi_0 + (1 - \frac{K}{k} \gamma_{11}) \xi_1 - \frac{K}{k} \gamma_{13} \xi_3 - \dots & \dots - \frac{K}{k} \delta_{11} \gamma_1 - \frac{K}{k} \delta_{13} \gamma_3 - \dots = \alpha_1 \\ - \frac{K}{k} \gamma_{30} \xi_0 - \frac{K}{k} \gamma_{31} \xi_1 + (1 - \frac{K}{k} \gamma_{33}) \xi_3 - \dots & \dots - \frac{K}{k} \delta_{31} \gamma_1 - \frac{K}{k} \delta_{33} \gamma_3 - \dots = \alpha_3 \\ \vdots & \vdots \\ - \frac{K}{k} \varepsilon_{10} \xi_0 - \frac{K}{k} \varepsilon_{11} \xi_1 - \frac{K}{k} \varepsilon_{13} \xi_3 - \dots & \dots + (1 - \frac{K}{k} \zeta_{11}) \gamma_1 - \frac{K}{k} \zeta_{13} \gamma_3 - \dots = \beta_1 \\ - \frac{K}{k} \varepsilon_{30} \xi_0 - \frac{K}{k} \varepsilon_{31} \xi_1 - \frac{K}{k} \varepsilon_{33} \xi_3 - \dots & \dots - \frac{K}{k} \zeta_{31} \gamma_1 + (1 - \frac{K}{k} \zeta_{33}) \gamma_3 - \dots = \beta_3 \\ \vdots & \vdots \end{aligned} \right\} \quad (61)$$

where

$$\gamma_{00}=0, \quad \gamma_{0m}=0, \quad \delta_{0m}=0 \quad (62)$$

$$\gamma_{0l} = -\frac{8}{[2]\pi^2} \frac{(-1)^{\frac{l-1}{2}}}{l} \sum_{n=2}^{\infty} (M_n \cos \varphi_n) \frac{n \sin n\theta_0}{\frac{l^2\pi^2}{\theta_0^2} - n^2} \quad (63)$$

$$\gamma_{lm} = -\frac{8}{[2]\pi\theta_0} \frac{m}{l} (-1)^{\frac{l+m-1}{2}} \sum_{n=2}^{\infty} (M_n \cos \varphi_n) \frac{n^2(1 + \cos n\theta_0)}{\left(\frac{m^2\pi^2}{\theta_0^2} - n^2 \right) \left(\frac{l^2\pi^2}{\theta_0^2} - n^2 \right)} \quad (64)$$

$$\delta_{lm} = \frac{8}{[2]\pi^2} \frac{(-1)^{\frac{l+m-1}{2}}}{l} \sum_{n=2}^{\infty} (M_n \sin \varphi_n) \frac{n^3(1+\cos n\theta_0)}{\left(\frac{m^2\pi^2}{\theta_0^2} - n^2\right) \left(\frac{l^2\pi^2}{\theta_0^2} - n^2\right)} \quad (65)$$

$$\varepsilon_{l0} = -\frac{8}{[2]\pi\theta_0} (-1)^{\frac{l-1}{2}} \sum_{n=2}^{\infty} (M_n \sin \varphi_n) \frac{\sin n\theta_0}{\frac{l^2\pi^2}{\theta_0^2} - n^2} \quad (66)$$

$$\varepsilon_{lm} = -\frac{8m}{[2]\theta_0^2} (-1)^{\frac{l+m-1}{2}} \sum_{n=2}^{\infty} (M_n \sin \varphi_n) \frac{n(1+\cos \theta_0)}{\left(\frac{m^2\pi^2}{\theta_0^2} - n^2\right) \left(\frac{l^2\pi^2}{\theta_0^2} - n^2\right)} \quad (67)$$

$$\zeta_{lm} = -\frac{8}{[2]\pi\theta_0} (-1)^{\frac{l+m-1}{2}} \sum_{n=2}^{\infty} (M_n \cos \varphi_n) \frac{n^2(1+\cos n\theta_0)}{\left(\frac{m^2\pi^2}{\theta_0^2} - n^2\right) \left(\frac{l^2\pi^2}{\theta_0^2} - n^2\right)} \quad (68)$$

$$\left. \begin{aligned} \alpha_0 &= \frac{H}{kA} \\ \alpha_l &= \frac{K}{k} (-1)^{\frac{l-1}{2}} \frac{\cos \frac{\theta_0}{2}}{\frac{l\pi}{4} \left(\frac{l^2\pi^2}{\theta_0^2} - 1\right)} \cos \gamma \\ \beta_l &= \frac{K}{k} (-1)^{\frac{l-1}{2}} \frac{\cos \frac{\theta_0}{2}}{\frac{\theta_0}{4} \left(\frac{l^2\pi^2}{\theta_0^2} - 1\right)} \sin \gamma \end{aligned} \right\} \quad (69)$$

The infinite series expressions for various coefficients γ_l , γ_{lm} , δ_{lm} , ε_{l0} , ε_{lm} and ζ_{lm} can be summed up into simple terms as follows.

By definition, we have

$$\begin{aligned} M_n e^{-j\varphi_n} &= M_n \cos \varphi_n - j M_n \sin \varphi_n = \frac{k F_1(jn\omega) F_2(jn\omega)}{G_1(jn\omega) G_2(jn\omega) + c_1 F_1(jn\omega) F_2(jn\omega)} \\ &= \frac{C(n^2) + jnD(n^2)}{A(n^2) + jnB(n^2)} = \frac{(AC - n^2 BD) + jn(BC + AD)}{A^2 - n^2 B^2}, \end{aligned} \quad (70)$$

where A , B , C and D are certain polynomials of n^2 , then $M_n \cos \varphi_n$ and $M_n \sin \varphi_n$ can be expanded into partial fractions of the following forms,

$$M_n \cos \varphi_n = \frac{AC - n^2 BD}{A^2 - n^2 B^2} = \sum_r \left\{ \frac{C_{r,p}}{(a_r - n^2)^p} + \frac{C_{r,p-1}}{(a_r - n^2)^{p-1}} + \dots + \frac{C_{r,1}}{a_r - n^2} \right\} \quad (71)$$

$$M_n \sin \varphi_n = -\frac{n(BC + AD)}{A^2 - n^2 B^2} = n \sum_r \left\{ \frac{d_{r,q}}{(b_r - n^2)^q} + \frac{d_{r,q-1}}{(b_r - n^2)^{q-1}} + \dots + \frac{d_{r,1}}{b_r - n^2} \right\} \quad (72)$$

where c_{rp} , d_{rq} , \dots , c_{r1} , d_{r1} , a_r and b_r may be complex numbers.

By utilizing these expressions and by expanding into partial fractions again, various infinite series expressing various coefficients (63) through (68) can be ultimately reduced to the sum of following two types of infinite series, which are

$$G_r(\mu) = \sum_{n=2}^{\infty} \frac{1 + \cos n\theta_0}{(\mu^2 - n^2)^r} \quad (73)$$

$$S_r(\mu) = \sum_{n=2}^{\infty} \frac{n \sin n\theta_0}{(\mu^2 - n^2)^r} \quad (74)$$

where μ is any complex number not equal to any integer, and r is any positive integer.

Again, owing to the relations that

$$S_r(\mu) = -\frac{d}{d\theta_0} G_r(\mu) \quad (75)$$

$$C_{r+1}(\mu) = -\frac{1}{r} \frac{d}{d\mu^2} C_r(\mu) \quad (76)$$

$$S_{r+1}(\mu) = -\frac{1}{r} \frac{d}{d\mu^2} S_r(\mu), \quad (77)$$

only the results for the following two series are necessary for calculating them. These are

$$\sum_{n=1,2,3,\dots} \frac{1 + \cos n\theta_0}{\mu^2 - n^2} = -\frac{1}{\mu^2} + \frac{\pi}{2\mu} \frac{\cos \mu(\pi - \theta_0) + \cos \mu\pi}{\sin \mu\pi}, \quad \mu \neq \pm n, \quad 0 \leq \theta_0 \leq 2\pi \quad (78)$$

$$\sum_{n=1,3,5,\dots} \frac{1 + \cos n\theta_0}{\mu^2 - n^2} = -\frac{\pi}{4\mu} \frac{\sin \mu\left(\frac{\pi}{2} - \theta_0\right) + \sin \mu \frac{\pi}{2}}{\cos \mu \frac{\pi}{2}}, \quad \mu \neq \pm n, \quad 0 \leq \theta_0 \leq \pi. \quad (79)$$

The summing up for various coefficients is actually performed for the case of the element with dead zone⁹⁾ and that with saturation characteristics¹⁰⁾.

Again, in this case, the set of solution (ξ_0, ξ_m, η_m) can be written as

$$\begin{pmatrix} \xi_0 \\ \xi_m \\ \eta_m \end{pmatrix} = \begin{pmatrix} \xi_0' \\ \xi_m' \\ \eta_m' \end{pmatrix} \cos \gamma + \begin{pmatrix} \xi_0'' \\ \xi_m'' \\ \eta_m'' \end{pmatrix} \sin \gamma + \begin{pmatrix} \xi_0''' \\ \xi_m''' \\ \eta_m''' \end{pmatrix} \frac{H}{kA} \quad (80)$$

where $(\xi_0', \xi_m', \eta_m')$, $(\xi_0'', \xi_m'', \eta_m'')$ and $(\xi_0''', \xi_m''', \eta_m''')$ are the solutions of (61), when $(\alpha_0, \alpha_l, \beta_l)$ of the right sides are replaced by

$$\begin{pmatrix} \alpha_0' \\ \alpha_l' \\ \beta_l' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{K}{k} (-1)^{\frac{l-1}{2}} \frac{\cos \frac{\theta_0}{2}}{\frac{l\pi}{4} \left(\frac{l^2 \pi^2}{\theta_0^2} - 1 \right)} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha_0'' \\ \alpha_l'' \\ \beta_l'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{K}{k} (-1)^{\frac{l-1}{2}} \frac{\cos \frac{\theta_0}{2}}{\frac{\theta_0}{4} \left(\frac{l^2 \pi^2}{\theta_0^2} - 1 \right)} \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha_0''' \\ \alpha_l''' \\ \beta_l''' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (81)$$

These sets of solutions $(\xi_0', \xi_m', \eta_m')$ etc are given as the functions of the input frequency ω and the nonlinear ratio $\theta_0/2\pi$ for the given system, and, introduced into (59), give the sets (x_0', x_n', y_n') etc.

The unique existence of the solution and the convergence to the exact value of the approximate solution are also assured in this case, similarly to the case of the direct method, as to be shown in the appendix.

As explained in paragraph 2, if we have determined the three sets of nondimensional coefficients (x_0', x_n', y_n') , (x_0'', x_n'', y_n'') and (x_0''', x_n''', y_n''') as the functions of the input frequency and the nonlinear ratio, our task of solving the problem is said to be completed. Namely, we can calculate the amplitude and the phase lag angle from (41) and (42) and the wave form from (26), as the angle γ is determined by

(42), and lastly the corresponding amplitude of the total input f from (43). Or, if the amplitude of the input is given at the outset, the corresponding nonlinear ratio can be determined by reversing the functional relation (43).

Usually, the convergency of the transformed series become very good, so that the number of equations to be solved for practical accuracy can be reduced to only small, three or so.

5. Conclusion

We have given a method of obtaining a perfect Fourier series solution for the periodic response to a sinusoidal input of the single-loop automatic control system including a piecewise-linear element. The piecewise-linear characteristics treated in this paper are such that the whole phase plane or phase space representing the dynamic states of the system is divided into two regions for the unsymmetrical case, or into three regions for the symmetrical case, in each of which the system is linear, and that the conditions of switching over between them may have

jump and hysteresis. And, further, we have confined ourselves to the case, when the number of switching points are two for the unsymmetrical case and four for the symmetrical case.

The brief outline of our procedures may be here once more given. We regard the nonlinear part of the output of the piecewise-linear element as if it were an input to the system from without, and obtain the formal solution for the linearized equation, which is deduced from the original nonlinear equation by assuming Fourier series expansion for this nonlinear part. The crux of our problem consists in the determination of these Fourier coefficients so as to make this formal solution satisfy the conditions of piecewise-linear characteristics and those of switching over. These requirements results in an infinite set of simultaneous linear equations for the nondimensionalized coefficients of this Fourier expansion as infinite number of unknowns.

The unique existence of solution for these equations and the convergence to the exact value of the approximative solutions obtained by discarding the equations and unknowns of higher orders are assured with discrete exceptional cases, which circumstances are similar to the case of finite number of equations and unknowns.

These equations can be solved directly or indirectly. The direct method has poor convergency, and is suitable for obtaining the general features of the response such as the amplitude and the phase lag angle of the fundamental harmonic. And the indirect method utilizes the appropriate series transformation for convergency improvement, and enables us to obtain its minute structure, such as the wave forms etc.

In principle, our method is not confined to the case of only one piecewise-linear element in the system, nor to the case of only two (three) regions composing the whole piecewise characteristics, nor to the case of only two (four) switching points between them. The superharmonic resonance belongs to these cases. And further, the subharmonic resonance may be attacked by assuming the Fourier expansion for the non-

linear part of the output of the piecewise-linear element, of the period several times that of the whole input, instead of the same assumed in this paper. But these problems are the object of future study.

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Appendix.

Existence and Convergency of Solution of an Infinite Set of Equations in Paragraph 3 and 4.

The infinite set of simultaneous linear equations determining the nondimensionalized Fourier coefficients for the nonlinear part of the output of the piecewise-linear element concerned, namely the equations (45) for the direct method or the equations (61) for the indirect method, takes the form

$$(E-A)z=c \quad (A.1)$$

In this $z(z_1, z_2, z_3, \dots)$ and $c(c_1, c_2, c_3, \dots)$ are two vectors in a vector space of infinite dimensions, and A and E are linear operators in this space, so that A is represented by the infinite matrix

$$A = (a_{ik}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (A.2)$$

and E is represented by the unit matrix,

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (A.3)$$

In fact, for the direct method,

$$z = (x_0, x_1, y_1, x_2, y_2, \dots) \quad (A.4)$$

$$c = (\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \dots) \quad (A.5)$$

and

$$A = \frac{K}{k} \begin{pmatrix} \gamma_{00} & \gamma_{01} & \delta_{01} & \gamma_{02} & \delta_{02} & \dots \\ \gamma_{10} & \gamma_{11} & \delta_{11} & \gamma_{12} & \delta_{12} & \dots \\ \varepsilon_{10} & \varepsilon_{11} & \zeta_{11} & \varepsilon_{12} & \zeta_{12} & \dots \\ \gamma_{20} & \gamma_{21} & \delta_{21} & \gamma_{22} & \delta_{22} & \dots \\ \varepsilon_{20} & \varepsilon_{21} & \zeta_{21} & \varepsilon_{22} & \zeta_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (A.6)$$

And, for the indirect method,

$$z = (\xi_0, \xi_1, \eta_1, \xi_2, \eta_2, \dots) \quad (A.7)$$

$$c = (\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \dots) \quad (A.8)$$

and

$$A = \frac{K}{k} \begin{pmatrix} \gamma_{00} & \gamma_{01} & \delta_{01} & \gamma_{02} & \delta_{02} & \dots \\ \gamma_{10} & \gamma_{11} & \delta_{11} & \gamma_{12} & \delta_{12} & \dots \\ \varepsilon_{10} & \varepsilon_{11} & \zeta_{11} & \varepsilon_{12} & \zeta_{12} & \dots \\ \gamma_{20} & \gamma_{21} & \delta_{21} & \gamma_{22} & \delta_{22} & \dots \\ \varepsilon_{20} & \varepsilon_{21} & \zeta_{21} & \varepsilon_{22} & \zeta_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (A.9)$$

For both cases, we can prove the convergence of the infinite sum

$$\|c\| = \sum_{i=1}^{\infty} |c_i|^2 = |c_1|^2 + |c_2|^2 + \dots, \quad (A.10)$$

so that the vector c belongs to a Hilbert space.

In the following, we use the notations A_n , P_n , Q_n and R_n for the four matrices, whose elements are partly equal to the original matrix A and partly put equal to zero as shown in Fig. 6, where the elements in the hatched domain are equal to those of A and the elements in white domain are put equal to zero. For

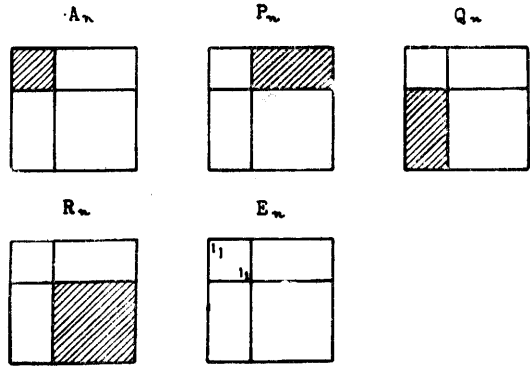


Fig. 6

example, A_n is the matrix, when in A all elements a_{ik} for $i > n$ and $k > n$ are put equal to zero. Lastly, E_n is the matrix, when an unit matrix is substituted for nonzero elements of P_n .

Now, we divide our equations into two parts, in one of which $i \leq n$ and in the other $i > n$, and we transform the latter into

$$z_j - \sum_{k=n+1}^{\infty} a_{jk} z_k = c_j + \sum_{k=1}^n a_{jk} z_k \quad (j = n+1, n+2, \dots) \quad (A.11)$$

If we also divide z into $z'(z_1, z_2, \dots, z_n, 0, 0, \dots)$ and $z''(0, 0, \dots, 0, z_{n+1}, z_{n+2}, \dots)$ and c into $c'(c_1, c_2, \dots, c_n, 0, 0, \dots)$ and $c''(0, 0, \dots, 0, c_{n+1}, c_{n+2}, \dots)$, this equation can be written as

$$(E - E_n - R_n)z'' = c'' + Q_n z' \quad (A.12)$$

When the double infinite series composed of the squares of the elements a_{ik} converges, namely when

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 \quad (A.13)$$

converges, the superior limit for R_n for n sufficiently great become less than 1, so that for any vector x in Hilbert space the following inequality is satisfied,

$$\|R_n x\| < \|x\| \quad (A.14)$$

In this case, $E - R_n$ has an inverse matrix, so that*

$$(E - R_n)^{-1} = E + R_n + R_n^2 + \dots \equiv E + B \quad (A.15)$$

and the unique solution of (A-12) can be given as

$$z'' = (E - E_n + B)c'' + (E - E_n + B)Q_n z' \quad (A.16)$$

On the other hand, the equations for $i \leq n$ are transformed into

$$\left. \begin{aligned} z_i - \sum_{k=1}^n a_{ik} z_k &= c_i + \sum_{k=n+1}^{\infty} a_{ik} z_k \\ (i=1, 2, \dots, n) \end{aligned} \right\} \quad (\text{A.17})$$

namely

$$(E_n - A_n)z' = c' + P_n z'' \quad (\text{A.18})$$

If we introduce z'' of (A.16) into this equation, the equations for z' become

$$C_n z' = Dc, \quad (\text{A.19})$$

where

$$\left. \begin{aligned} C_n &= E_n - A_n - P_n(E+B)Q_n \\ D &= E_n + P_n(E+B) \end{aligned} \right\} \quad (\text{A.20})$$

Thus we have eliminated z_{n+1}, z_{n+2}, \dots from the first n equations by utilizing the other equations for which $i > n$. This possibility of elimination originates from the existence of the inverse matrix to $E - R_n$, and corresponds to that in the case of finite number of equations enabled by nonvanishing of the determinant of the matrix,

$$|E - R_n| \neq 0 \quad (\text{A.21})$$

The resulting equations after the elimination are finite in number, so the ordinary rule for the linear equations of finite number can be applied.

At this point, we distinguish between the following two cases.

$$(1) \quad |C_n| \neq 0$$

In this case, C_n^{-1} , the inverse matrix of C_n exists, and the equations (A.19) allow an unique solution, that is

$$z' = C_n^{-1} Dc \quad (\text{A.22})$$

and introducing this into (A.16), we obtain for z''

$$z'' = \{E - E_n + B + (E+B)Q_n C_n^{-1} D\} c \quad (\text{A.23})$$

and lastly from these for z

$$\begin{aligned} z = z' + z'' &= (E - E_n + B + C_n^{-1} D + Q_n C_n^{-1} D \\ &\quad + B Q_n C_n^{-1} D) c \end{aligned} \quad (\text{A.24})$$

This is the unique solution in this case, and the vector z belongs to Hilbert space.

$$(2) \quad |C_n| = 0$$

In this case, as the theory of linear equations shows, the solution becomes impossible, except when $c(c_1, c_2, \dots)$ satisfies certain conditions, and when these are satisfied the solution becomes indeterminate.

Namely, if we define a vector d by

$$d(d_1, d_2, \dots, d_n, 0, 0, \dots) = Dc \quad (\text{A.25})$$

and write the transposed matrix of C_n by C_n' , and m linearly independent solutions of the equations

$$C_n' y = 0 \quad (\text{A.26})$$

by $y^{(s)}$, so that

$$\left. \begin{aligned} y^{(s)} &= (y_1^{(s)}, y_2^{(s)}, \dots, y_n^{(s)}), \\ (s=1, 2, 3, \dots, m, n \geq m \geq 1) \end{aligned} \right\} \quad (\text{A.27})$$

the necessary and sufficient conditions for the existence of the solution for (A.19) are the fulfillment of m homogeneous equations, which are

$$\left. \begin{aligned} (y^{(s)} d) &= y_1^{(s)} d_1 + y_2^{(s)} d_2 + \dots \\ &\quad + y_n^{(s)} d_n = 0 \\ (s=1, 2, \dots, m) \end{aligned} \right\} \quad (\text{A.28})$$

If these are fulfilled, the solution z' can be written in the following form

$$z' = k_1 z'^{(1)} + k_2 z'^{(2)} + \dots + k_m z'^{(m)} + b \quad (\text{A.29})$$

where k_1, k_2, \dots, k_m are m arbitrary constants and $z'^{(1)}, z'^{(2)}, \dots, z'^{(m)}$ are m independent solutions of the homogeneous equation,

$$C_n z' = 0, \quad (\text{A.30})$$

and $b(b_1, b_2, \dots, b_n)$ is any particular solution of the original nonhomogeneous equation (A.19), which is assured by (A.28)

This case is left as the object of the future study, though it is very interesting as it is closely related to the problem of the superharmonic resonance.

We return now to the first case, where $|C_n| \neq 0$, and the unique solution can be exactly obtained by calculating (A.24). But in praxis, the summing up of the infinite series compose of matrices as individual terms is very laborious up to almost impracticable, so we are obliged to resort to the approximative solution by discarding the equations and unknowns of higher orders. This approximative method is called as the method of curtailment.

Namely, we define the approximate solution of n -th order $z^{(n)}$ by

$$(E - A_n)z^{(n)} = c \quad (\text{A.31})$$

which is written in full as

$$\left. \begin{aligned} z_i^{(n)} - \sum_{k=1}^n a_{ik} z_k^{(n)} &= c_i, \quad i \leq n \\ z_j^{(n)} &= z_j, \quad j > n \end{aligned} \right\} \quad (\text{A.32})$$

And we can prove in our case that this approximate solution converges to the exact value as

n tends to infinity.

$$\lim_{n \rightarrow \infty} \|z^{(n)} - z\| = 0, \text{ so } \lim_{n \rightarrow \infty} z^{(n)} = z \quad (\text{A.33})$$

This is the so-called principle of curtailment, and is realized when A is completely continuous^{*)}, which condition results further from the convergency of the double series $\sum_{i,k=1}^{\infty} |a_{ik}|^2$ in our case.

Thus, the unique existence of solution and the convergence of the approximate solution by curtailment are both assured by the condition that the double series

$$\left. \begin{aligned} |\gamma_{0n}|^2 &= O\left(\frac{1}{n^4}\right), \quad |\gamma_{mn}|^2 = O\left(\frac{1}{m^4}\right) \\ |\gamma_{mn}|^2 &= O\left\{\frac{1}{(m+n)^2 n^4}\right\} + O\left\{\frac{1}{(m-n)^2 n^4}\right\} + O\left\{\frac{1}{(m^2-n^2)n^4}\right\} + O\left\{\frac{1}{m^2 n^4}\right\} \\ |\delta_{0n}|^2 &= O\left(\frac{1}{n^2}\right), \quad |\delta_{mn}|^2 = O\left(\frac{1}{n^2}\right) \\ |\delta_{mn}|^2 &= O\left\{\frac{1}{(m+n)^2 n^2}\right\} + O\left\{\frac{1}{(m-n)^2 n^2}\right\} + O\left\{\frac{1}{(m^2-n^2)n^2}\right\} + O\left(\frac{1}{m^2 n^2}\right) \\ |\zeta_{nn}|^2 &= O\left(\frac{1}{n^4}\right), \quad |\zeta_{mn}|^2 = O\left\{\frac{1}{(m+n)^2 n^4}\right\} + O\left\{\frac{1}{(m-n)^2 n^4}\right\} + O\left\{\frac{1}{(m^2-n^2)n^4}\right\} \end{aligned} \right\} \quad (\text{A.34})$$

where

$$z_{m,n} = O\{f(m, n)\} \quad (\text{A.35})$$

means that we can choose a positive constant K for any given values m_0 and n_0 such that

$$|z_{mn}/f(m, n)| < K \text{ as } m > m_0 \text{ and } n > n_0. \quad (\text{A.36})$$

Because

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 = \frac{K}{k} \left\{ \sum_{n=1}^{\infty} |\gamma_{0n}|^2 + \sum_{m=1}^{\infty} |\gamma_{mn}|^2 + \sum_{m,n=1}^{\infty} |\gamma_{mn}|^2 + \sum_{n=1}^{\infty} |\delta_{0n}|^2 + \sum_{m,n=1}^{\infty} |\delta_{mn}|^2 + \sum_{m=1}^{\infty} |\varepsilon_{mn}|^2 + \sum_{m,n=1}^{\infty} |\varepsilon_{mn}|^2 + \sum_{m=1}^{\infty} |\zeta_{mm}|^2 + \sum_{m,n=1}^{\infty} |\zeta_{mn}|^2 \right\} \quad (\text{A.37})$$

where $\sum_{m,n=1}^{\infty}$ means the sum for any positive combinations of all positive integers m and n except for $m=n$, and because each of the infinite sums

$$\sum_{n=1}^{\infty} \frac{1}{n^4}, \quad \sum_{m=1}^{\infty} \frac{1}{m^4}, \quad \sum_{m,n=1}^{\infty} \frac{1}{(m+n)^2 n^4}, \quad \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{m,n=1}^{\infty} \frac{1}{(m+n)^2 n^2}, \quad \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^2}$$

is convergent, our double series $\sum_{i,k=1}^{\infty} |a_{ik}|^2$ is convergent, if each of the series

$$\sum_{m,n=1}^{\infty} \frac{1}{(m-n)^2 n^4}, \quad \sum_{m,n=1}^3 \frac{1}{(m^2-n^2)n^4}, \quad \sum_{m,n=1}^{\infty} \frac{1}{(m-n)^2 n^2}, \quad \sum_{m,n=1}^{\infty} \frac{1}{(m^2-n^2)n^2}$$

converges.

But the last two converge if the former two do, and, as

$$\sum_{m,n=1}^{\infty} \frac{1}{(m-n)^2 n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{\infty} \frac{1}{(m-n)^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{\infty} \frac{2}{m^2} \quad (\text{A.38})$$

and

$$\sum_{m,n=1}^{\infty} \frac{1}{(m^2-n^2)n^2} \leq \sum_{m,n=1}^{\infty} \frac{1}{2m(m-n)n^2} + \sum_{m,n=1}^{\infty} \frac{1}{2m(m+n)n^2}$$

converges. That this actually converges in our cases, is easily proved both for the direct and indirect method by considering the orders of magnitude as i and k tend to infinity, of the given expressions for various coefficients in paragraph 3 and 4.

For the direct method, we have from (46), (47), (48), (49), (71) and (72), for $a_r \neq n^2$ and $b_r = n^2$ ($n=2, 3, 4, \dots$),

*) Riesz, F., Les Systemes d'equations lineaires a une infinite d'inconnues (Gauthier-Villars) 1952.

$$\begin{aligned}
&\leq \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \frac{1}{2m} \left\{ \frac{1}{m^2} \left(\frac{1}{m-n} + \frac{1}{n} \right) + \frac{1}{m} \frac{1}{n^2} \right\} + \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \frac{1}{2m(m-n)n^2} + \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^2} \\
&= \sum_{m=1}^{\infty} \left\{ \frac{1}{2m} \frac{2(m-1)}{m^2} + \frac{1}{2m^2} \sum_{n=1}^{m-1} \frac{1}{n^2} \right\} + \sum_{m=1}^{\infty} \frac{1}{2m} \sum_{k=1}^{\infty} \frac{1}{(m+k)^2} + \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^2} \\
&\leq \sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{m=1}^{\infty} \frac{1}{2m^2} \int_{\frac{1}{2}}^{m-\frac{1}{2}} \frac{dx}{x^2} + \sum_{m=1}^{\infty} \frac{1}{2m} \int_m^{\infty} \frac{dx}{x^2} + \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^2} \\
&\leq \frac{5}{2} \sum_{m=1}^{\infty} \frac{1}{m^2} + \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^2}, \tag{A.39}
\end{aligned}$$

the former two do actually converge.

Thus we have proved the convergence of the double series $\sum_{i,k=1}^{\infty} |a_{jk}|^2$ when $a_r \neq n^2$ and $b_r \neq n^2$ ($n=2, 3, \dots$) for the case of the direct method.

Secondly, for the case of the indirect method, we obtain, from (62) through (68), (71) and (72), the order of magnitude for various coefficients. For example, for γ_n , we have

$$\begin{aligned}
\gamma_n &= -\frac{8}{[2]} \frac{(-1)^{\frac{l-1}{2}}}{\pi^2 l} \sum_{n=2}^{\infty} (M_n \cos \varphi_n) \frac{n \sin n \theta_0}{\frac{l^2 \pi^2}{\theta_0^2} - n^2} \\
&= -\frac{8}{[2]} \frac{(-1)^{\frac{l-1}{2}}}{\pi^2 l} \sum_r \sum_{s=1}^p \frac{c_{r,s}}{(a_r - n^2)^s} \frac{n \sin n \theta_0}{\left(\frac{l^2 \pi^2}{\theta_0^2} - n^2 \right)} \\
&= -\frac{8}{[2]} \frac{(-1)^{\frac{l-1}{2}}}{\pi^2 l} \sum_r \sum_{s=1}^p \frac{(-1)^{s-1}}{(s-1)!} \left(\frac{d}{da_r} \right)^{s-1} \sum_{n=2}^{\infty} \frac{n \sin n \theta_0}{(a_r - n^2) \left(\frac{l^2 \pi^2}{\theta_0^2} - n^2 \right)} \tag{A.40}
\end{aligned}$$

and when $a_r \neq n^2$ ($n=2, 3, 4, \dots$),

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{n \sin n \theta_0}{(a_r - n^2) \left(\frac{l^2 \pi^2}{\theta_0^2} - n^2 \right)} &= \frac{1}{a_r - \frac{l^2 \pi^2}{\theta_0^2}} \sum_{n=1}^{\infty} \left\{ \frac{n \sin n \theta_0}{\frac{l^2 \pi^2}{\theta_0^2} - n^2} - \frac{n \sin n \theta_0}{a_r - n^2} \right\} - \frac{\sin \theta_0}{(a_r - 1) \left(\frac{l^2 \pi^2}{\theta_0^2} - 1 \right)} \\
&= \frac{1}{\frac{l^2 \pi^2}{\theta_0^2} - a_r} \frac{\pi}{2} \frac{\sin \sqrt{a_r} \pi + \sin \sqrt{a_r} (\pi - \theta_0)}{\sin \sqrt{a_r} \pi} - \frac{\sin \theta_0}{(a_r - 1) \left(\frac{l^2 \pi^2}{\theta_0^2} - 1 \right)} \tag{A.41}
\end{aligned}$$

for the unsymmetrical case, and

$$= \frac{1}{\frac{l^2 \pi^2}{\theta_0^2} - a_r} \frac{\pi}{4} \frac{\cos \sqrt{a_r} \frac{\pi}{2} + \cos \sqrt{a_r} \left(\frac{\pi}{2} - \theta_0 \right)}{\cos \sqrt{a_r} \frac{\pi}{2}} - \frac{\sin \theta_0}{(a_r - 1) \left(\frac{l^2 \pi^2}{\theta_0^2} - 1 \right)} \tag{A.42}$$

for the symmetrical case. So, the order of magnitude of $|\gamma_n|$ is given as

$$|\gamma_n| = O\left(\frac{1}{l^2}\right)$$

Similarly, for $a_r \neq n^2$ and $b_r \neq n^2$ ($n=2, 3, 4, \dots$), we have

$$\begin{aligned}
|\gamma_u| &= O\left(\frac{1}{l^2}\right), & |\gamma_{lm}| &= O\left(\frac{1}{l^3 m}\right) \\
|\delta_u| &= O\left(\frac{1}{l}\right), & |\delta_{lm}| &= O\left(\frac{1}{l^3 m}\right) \\
|\varepsilon_u| &= O\left(\frac{1}{l^2}\right), & |\varepsilon_u| &= O\left(\frac{1}{l^2}\right), & |\varepsilon_{lm}| &= O\left(\frac{1}{l^2 m^2}\right) \\
|\zeta_u| &= O\left(\frac{1}{l^2}\right), & |\zeta_{lm}| &= O\left(\frac{1}{l^2 m^2}\right)
\end{aligned}$$

And because

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 = \frac{K}{k} \left\{ \sum_{l=1,3,5\dots} |\gamma_{l0}|^2 + \sum_{l=1,3,5\dots} |\gamma_{lu}|^2 + \sum'_{l,m=1,3,5\dots} |\gamma_{lm}|^2 + \sum_{l=1,3,5\dots} |\delta_{lu}|^2 + \sum'_{l,m=1,3,5\dots} |\delta_{lm}|^2 \right. \\ \left. + \sum_{l=1,3,5\dots} |\varepsilon_{l0}|^2 + \sum_{l=1,3,5\dots} |\varepsilon_{lu}|^2 + \sum'_{l,m=1,3,5\dots} |\varepsilon_{lm}|^2 + \sum_{l=1,3,5\dots} |\hat{\varepsilon}_{lu}|^2 + \sum_{l,m=1,3,5\dots} |\hat{\varepsilon}_{lm}|^2 \right\}, \quad (\text{A.45})$$

our double series $\sum_{i,k=1}^{\infty} |a_{ik}|^2$ actually converges, except when

$$a_r = n^2, \quad b_r = n^2 \quad (n=2, 3, 4, \dots) \quad (\text{A. 46})$$

The exceptional cases (A.46) for both the direct method and the indirect method belong to the cases of superharmonic resonance and are left for the future study.