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Abstract. In this paper, we investigate the asymptotic behavior of solutions to the Cauchy problem for the generalized Korteweg-de Vries-Burgers equation, where the far field states are prescribed. When the corresponding Riemann problem for the hyperbolic part admits a Riemann solution which consists of single rarefaction wave, we expect that the unique global in time solution to the Cauchy problem tends toward the rarefaction wave as time goes to infinity. We introduce how to obtain the global asymptotic stability of the rarefaction wave.

要旨：本論文においては、一般化 Korteweg-de Vries-Burgers 方程式の遠方条件付き Cauchy 問題の解の漸近挙動が考察されている。対応する双曲型保存則の Riemann 問題の解が単独の希薄波によって構成される場合、この Cauchy 問題の一意的時間大域解は希薄波へ時間と共に漸近すると予想される。この希薄波の大域漸近安定性が如何にして得られるかを紹介する。

1. Introduction and main theorems.

We consider the asymptotic behavior of solutions to the Cauchy problem for the generalized Korteweg-de Vries Burgers equation

$$\begin{cases} \partial_t u + \partial_x (f(u) - \mu \partial_x u + \delta \partial_x^2 u) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) \rightarrow u_{\pm} & (x \rightarrow \pm\infty), \end{cases} \quad (1.1)$$

where, $u(t, x)$ is the unknown function of $t > 0$ and $x \in \mathbb{R}$, the so-called conserved quantity,

$$f(u) - \mu \partial_x u + \delta \partial_x^2 u \quad (\mu > 0, \delta \in \mathbb{R})$$

is the total flux (that is, the functions, $f(u)$ and $\mu \partial_x u$ stand for the convective flux, viscous/diffusive one and dispersive one, respectively), u_0 is the initial data, and $u_{\pm} \in \mathbb{R}$ are the prescribed far field states.

It is noted that if $\delta = 0$, then the equation in (1.1) becomes the generalized viscous Burgers equation

$$\partial_t u + \partial_x (f(u)) = \mu \partial_x^2 u,$$

if $\delta = 0$ and $f(u) = u^2/2$, then the following equation is the so-called viscous Burgers equation

$$\partial_t u + u \partial_x u = \mu \partial_x^2 u,$$

and if $\delta \neq 0$, $\delta \mapsto -\delta$ and $\mu = 0$, then does the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x (f(u)) = \delta \partial_x^3 u.$$

Here, we note that the viscous Burgers equation and its generalization of the convective flux or viscous/diffusive one are also said to be viscous/diffusive conservation law. Also the Korteweg-de Vries equation and Korteweg-de Vries-Burgers equation (and their generalization of the fluxes) should be categorized as diffusive conservation law and diffusive dispersive conservation law, respectively.

We are interested in the global asymptotic stability of the rarefaction wave to (1.1), therefore deal with the case where the smooth convective flux f is fully convex, that is,

$$f''(u) > 0 \quad (u \in \mathbb{R}) \quad (1.2)$$

and $u_- < u_+$. Then, since the corresponding Riemann problem

$$\begin{cases} \partial_t u + \partial_x (f(u)) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0^R(x) := \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0), \end{cases} \end{cases} \quad (1.3)$$

turns out to admit a single rarefaction wave solution, we expect that the solution of the Cauchy problem (1.1) tends toward the rarefaction wave as time goes to infinity. Here, the rarefaction wave connecting u_- to u_+ is given by

$$u^r \left(\frac{x}{t}; u_-, u_+ \right) = \begin{cases} u_- & (x \leq f'(u_-)t), \\ (f')^{-1} \left(\frac{x}{t} \right) & (f'(u_-)t \leq x \leq f'(u_+)t), \\ u_+ & (x \geq f'(u_+)t). \end{cases} \quad (1.4)$$

We also expect that if $u_- = u_+ =: \tilde{u}$, then the solution of the Cauchy problem (1.1) tends toward the constant state \tilde{u} as time goes to infinity. In fact, there are many results concerning with the rarefaction stabilities (see [23, 24, 36, 41] for shock and the other stabilities (such as some multiwave pattern), see [4, 10, 12, 13, 15, 19, 21, 22, 26, 27, 33, 35, 40, 43, 45], cf. [5, 15, 16, 20, 31, 44]). In particular, Il'in-Oleĭnik [17] considered the following Cauchy problem for the generalized viscous Burgers equation

$$\begin{cases} \partial_t u + \partial_x (f(u) - \mu \partial_x u) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) \rightarrow u_{\pm} & (x \rightarrow \pm\infty), \end{cases}$$

and obtained the global stability of single rarefaction wave (and single shock wave). Hattori-Nishihara [14] also obtained the pointwise and time-decay estimates of the difference $|u - u^r|$. Harabetian [11] further considered the following rarefaction problem for a quasilinear parabolic equations

$$\begin{cases} \partial_t u + \partial_x (f(u) - A'(u) \partial_x u) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) \rightarrow u_{\pm} & (x \rightarrow \pm\infty), \end{cases}$$

where $A'(u) \geq 0$ ($u \in \mathbb{R}$), and obtained the precise time-decay estimates of global stability of single rarefaction wave with the aid of the arguments on monotone semigroups by Osher-Ralston [30]. For the following Cauchy problem of the Matsumura-Nishihara model

$$\begin{cases} \partial_t u + \partial_x (f(u) - \mu |\partial_x u|^{p-1} \partial_x u) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) \rightarrow u_{\pm} & (x \rightarrow \pm\infty), \end{cases}$$

where $p > 1$ and the viscosity $\mu |\partial_x u|^{p-1} \partial_x u$ is the so-called Ostwald-de Waele-type viscosity (which is a typical example for the non-Newtonian viscosity), Matsumura-Nishihara [25] first investigated and proved the global stability of single rarefaction wave by using the technical energy method. Yoshida [36] further obtained its precise time-decay estimates by using the time-weighted energy method (for the stabilities of the multiwave pattern, see [37-39]). Furthermore, Matsumura-Yoshida [28] considered the following Cauchy problem of the non-Newtonian viscous conservation law

$$\begin{cases} \partial_t u + \partial_x (f(u) - \sigma(\partial_x u)) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) \rightarrow u_{\pm} & (x \rightarrow \pm\infty), \end{cases}$$

where viscosity function σ satisfies the conditions

$$\begin{cases} \sigma(0) = 0, \quad \sigma'(v) > 0 \quad (v \in \mathbb{R}), \\ |\sigma(v)| \sim |v|^p, \quad |\sigma'(v)| \sim |v|^{p-1} \quad (|v| \rightarrow \infty), \end{cases}$$

and obtained the global stability of single rarefaction wave for the case $0 < p < 3/7$. Recently, Yoshida [42] obtained this rarefaction stability for more general case $0 < p < 1/3$ and its precise time-decay estimates. Egorova-Grunert-Teschl [8] and Egorova-Teschl [9] investigated the following rarefaction problem for the Korteweg-de Vries equation

$$\begin{cases} \partial_t u - 6u \partial_x u = -\partial_x^3 u & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) \begin{cases} \rightarrow 0 & (x \rightarrow \infty), \\ \rightarrow u_- > 0 & (x \rightarrow -\infty), \end{cases} \end{cases}$$

and showed the existence and uniqueness of the classical solution in some class. Andreiev-Egorova-Lange-Teschl [1] further obtained the valid asymptotic formula of the solution. On the other hand, Wang-Zhu [34] investigated the rarefaction problem for the Korteweg-de Vries-Burgers equation (1.1) and obtained the local stability of the rarefaction wave under the conditions

$$\begin{cases} \exists \alpha > 0; \forall u \in \mathbb{R}, f''(u) \geq \alpha, \\ \exists \eta > 0; \|u_0(x) - U^r(0, x)\|_{H^1}^2 + h(u_+ - u_-) \leq \eta, \quad \lim_{s \rightarrow 0+0} h(s) = 0, \end{cases}$$

where U^r is a smooth approximation for u^r which is defined by (2.1) in Section 2. Duan-Zhao [7] also obtained the global stability of the rarefaction wave under the following condition

$$|f'(u)| \leq O(1) (1 + |u|^p) \quad (0 < p < 2)$$

or

$$|f''(u)| \leq O(1)(1 + |u|)$$

(for the stability of a travelling wave and its time-decay properties, see [2,3,29]). For the following Cauchy problem of the generalized Korteweg-de Vries Burgers-Kuramoto equation

$$\begin{cases} \partial_t u + \partial_x (f(u) - \mu \partial_x u + \delta \partial_x^2 u + \nu \partial_x^3 u) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) \rightarrow u_{\pm} & (x \rightarrow \pm\infty), \end{cases}$$

where $\nu > 0$, Ruan-Gao-Chen [32] obtained the local stability of the rarefaction wave. Duan-Fan-Kim-Xie [6] also obtained the global stability of the rarefaction wave under the following condition

$$|f'(u)| \leq O(1)(1 + |u|^p) \quad (0 < p < 4), \quad |f''(u)| \leq O(1)(1 + |u|^3)$$

or

$$\exists C > 0; 16 \left(C(1 + \nu + \|u_0 - U^r(0)\|_{L^2}^2) \right)^6 \sup_{u \in \mathbb{R}} \left| \frac{d^7 f}{du^7}(u) \right| < \frac{\nu^2}{8}.$$

Yoshida [41] further obtained the same stability as that in [6] for more general cases as follows.

$$|f'(u)| \leq O(1)(1 + |u|^p) \quad (0 \leq p \leq 6), \quad |f''(u)| \leq O(1)(1 + |u|^q) \quad (1 \leq q \leq 5)$$

or

$$f \in \bigcup_{n=2}^7 \left\{ g \in C^n(\mathbb{R}) \left| \sup_{u \in \mathbb{R}} \left| \frac{d^n g}{du^n}(u) \right| < \infty \right. \right\} \cap C^5(\mathbb{R}).$$

Our main results of the present paper are as follows.

Theorem 1.1. Assume the far field states u_{\pm} satisfy $u_- = u_+ = \tilde{u}$, the convective flux $f \in C^1(\mathbb{R})$ satisfy

$$|f'(u)| \leq O(1)(1 + |u|^p) \quad (0 \leq p \leq 2). \quad (1.5)$$

Further assume the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in L^2$. When $p = 0$ or $p = 2$, then the Cauchy problem (1.1) has a unique global in time solution u satisfying

$$\begin{cases} u - \tilde{u} \in C^0([0, \infty); H^1), \\ \partial_x u \in L^2(0, \infty; H^1), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |u(t, x) - \tilde{u}| = 0.$$

Theorem 1.2. Assume the far field states u_{\pm} satisfy $u_- = u_+ = \tilde{u}$, the convective flux $f \in C^2(\mathbb{R})$ satisfy

$$|f''(u)| \leq O(1)(1 + |u|^q) \quad (0 \leq q \leq 1). \quad (1.6)$$

Further assume the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in L^2$. Then the same result as in Theorem 1.1 holds true.

Theorem 1.3. Assume the far field states u_{\pm} satisfy $u_- < u_+$, the convective flux $f \in C^4(\mathbb{R})$ satisfy (1.2) and (1.5). Further assume the initial data satisfy $u_0 - u_0^R \in L^2$ and $\partial_x u_0 \in L^2$. Then the Cauchy problem (1.1) has a unique global in time solution u satisfying

$$\begin{cases} u - u_0^R \in C^0([0, \infty); H^1), \\ \partial_x u \in L_{\text{loc}}^2(0, \infty; H^1), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| u(t, x) - u^r \left(\frac{x}{t}; u_-, u_+ \right) \right| = 0.$$

Theorem 1.4. Assume the far field states u_{\pm} satisfy $u_- < u_+$, the sufficiently smooth convective flux satisfy (1.2) and (1.6). Further assume the initial data satisfy $u_0 - u_0^R \in L^2$ and $\partial_x u_0 \in L^2$. Then the same result as in Theorem 1.1 holds true.

For the generalized Korteweg-de Vries Burgers-Kuramoto equation, we can further obtain the next theorem. Because the proof is similarly given as Theorem 1.4, we state here only.

Theorem 1.5. Assume the far field states u_{\pm} satisfy $u_- < u_+$, the convective flux $f \in C^5(\mathbb{R})$ satisfy (1.2) satisfy

$$|f''(v)| \leq O(1) (1 + |v|^q) \quad (0 \leq q \leq 5). \quad (1.7)$$

Further assume the initial data satisfy $v_0 - u_0^R \in L^2$ and $\partial_x v_0 \in L^2$. Then the Cauchy problem for the generalized Korteweg-de Vries Burgers-Kuramoto equation

$$\begin{cases} \partial_t v + \partial_x (f(v) - \mu \partial_x v + \delta \partial_x^2 v + \nu \partial_x^3 v) = 0 & (t > 0, x \in \mathbb{R}), \\ v(0, x) = v_0(x) \rightarrow u_{\pm} & (x \rightarrow \pm\infty), \end{cases}$$

where $\nu > 0$, has a unique global in time solution u satisfying

$$\begin{cases} v - u_0^R \in C^0([0, \infty); H^2), \\ \partial_x v \in L_{\text{loc}}^2(0, \infty; H^2), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| v(t, x) - u^r \left(\frac{x}{t}; u_-, u_+ \right) \right| = 0.$$

We should emphasize that the Theorem 1.3 with $0 < p < 2$ and theorem 1.4 with $q = 1$ had been obtained by Duan-Zhao [6], the Theorems 1.1-1.4 with $0 \leq p \leq 2$ and $0 \leq q \leq 1$ had been stated in Yoshida [41] without the proofs and Theorem 1.5 with $1 \leq q \leq 5$ had been obtained by [41]. Because the proofs of Theorems 1.1 and 1.2 are easier than those of Theorems 1.3 and 1.4, we only treat Theorems 1.3 and 1.4.

This paper is organized as follows. In Section 2, we prepare the basic properties of the rarefaction wave. In Section 3, we reformulate the rarefaction problem in terms of the deviation from the asymptotic state. Also, in order to show the global existence and asymptotic behavior of solution to the reformulated problem, we give the local existence theorem and the *a priori* estimates. Next in Section 4, we remark some uniform estimates by

applying theorems in Section 3. In order to show the asymptotics, we establish the *a priori* estimates by using the technical energy method in Section 5.

Some Notation. We denote by C generic positive constants unless they need to be distinguished. In particular, use $C_{\alpha, \beta, \dots}$ when we emphasize the dependency on α, β, \dots .

For function spaces, $L^p = L^p(\mathbb{R})$ and $H^k = H^k(\mathbb{R})$ denote the usual Lebesgue space and k -th order Sobolev space on the whole space \mathbb{R} with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$, respectively.

2. Approximation for the rarefaction wave.

In this section, we prepare the several lemmas concerning with the basic properties of the rarefaction wave for accomplishing the proof of the main Theorems 1.3-1.4. Since the rarefaction wave u^r , (1.4), is not smooth enough, we need some smooth approximated one.

We use the well-known approximation for the rarefaction wave $u^r(x/t; u_-, u_+)$ as follows.

$$U^r = U^r(t, x; q, u_-, u_+) := (\lambda)^{-1}(w(t, x; q, \lambda_-, \lambda_+)). \quad (2.1)$$

Here $\lambda(u) := f'(u)$, $\lambda_{\pm} := \lambda(u_{\pm}) = f'(u_{\pm})$ and

$$w = w(t, x; q, w_-, w_+) \in C^\infty([0, \infty) \times \mathbb{R})$$

is the unique classical solution to the following Cauchy problem for the non-viscous Burgers equation as

$$\begin{cases} \partial_t w + \partial_x \left(\frac{1}{2} w^2 \right) = 0 & (t > 0, x \in \mathbb{R}), \\ w(0, x) = w_0(x) := \frac{w_- + w_+}{2} + \frac{w_+ - w_-}{2} K_q \int_0^x \frac{dy}{(1+y^2)^q} & (x \in \mathbb{R}), \end{cases} \quad (2.2)$$

where K_q is a positive constant such that

$$K_q \int_0^\infty \frac{dy}{(1+y^2)^q} = 1 \quad \left(q > \frac{1}{2} \right).$$

By the method of characteristics, we can get the following formula.

$$\begin{cases} w(t, x) = w_0(x_0(t, x)) = \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} K_q \int_0^{x_0(t, x)} \frac{dy}{(1+y^2)^q}, \\ x = x_0(t, x) + w_0(x_0(t, x)) t. \end{cases} \quad (2.3)$$

From (2.3), we can obtain the properties of $w(t, x; w_-, w_+)$ in the next lemma.

Lemma 2.1. Assume that the far field states satisfy $w_- < w_+$. Then the classical solution $w(t, x)$ to (2.2) satisfies the following properties.

- (1) $w_- < w(t, x) < w_+$ and $\partial_x w(t, x) > 0$ ($t > 0, x \in \mathbb{R}$).
- (2) For any $r \in [1, \infty]$, there exists a positive constant $C_{w_{\pm}, q, r}$ such that

$$\begin{aligned}\|\partial_x w(t)\|_{L^r} &\leq C_{w_\pm, q, r} (1+t)^{-1+\frac{1}{r}} \quad (t \geq 0), \\ \|\partial_x^2 w(t)\|_{L^r} &\leq C_{w_\pm, q, r} (1+t)^{-1-\frac{1}{2q}(1-\frac{1}{r})} \quad (t \geq 0), \\ \|\partial_x^3 w(t)\|_{L^r} &\leq C_{w_\pm, q, r} (1+t)^{-1-\frac{1}{2q}(2-\frac{1}{r})} \quad (t \geq 0).\end{aligned}$$

$$(3) \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| w(t, x) - w^r\left(\frac{x}{t}\right) \right| = 0.$$

Noting the assumption of the convective flux f and using Lemma 2.1, we can obtain the properties of the smooth approximation U^r defined by (2.1) in the next lemma.

Lemma 2.2. *Assume that the far field states satisfy $u_- < u_+$, and the flux function $f \in C^4(\mathbb{R})$, $f''(u) > 0$ ($u \in [u_-, u_+]$). Then we have the following properties.*

(1) $U^r(t, x)$ defined by (2.1) is the unique smooth global solution in space-time of the Cauchy problem

$$\begin{cases} \partial_t U^r + \partial_x (f(U^r)) = 0 & (t > 0, x \in \mathbb{R}), \\ U^r(0, x) = (\lambda)^{-1} \left(\frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} K_q \int_0^x \frac{dy}{(1+y^2)^q} \right) & (x \in \mathbb{R}), \\ \lim_{x \rightarrow \pm\infty} U^r(t, x) = u_\pm & (t \geq 0). \end{cases}$$

(2) $u_- < U^r(t, x) < u_+$ and $\partial_x U^r(t, x) > 0$ ($t > 0, x \in \mathbb{R}$).

(3) For any $r \in [1, \infty]$, there exists a positive constant $C_{\lambda_\pm, r}$ such that

$$\begin{aligned}\|\partial_x U^r(t)\|_{L^r} &\leq C_{q, r} (1+t)^{-1+\frac{1}{r}} \quad (t \geq 0), \\ \|\partial_x^2 U^r(t)\|_{L^r} &\leq C_{q, r} (1+t)^{-1-\frac{1}{2q}(1-\frac{1}{r})} \quad (t \geq 0), \\ \|\partial_x^3 U^r(t)\|_{L^r} &\leq C_{q, r} (1+t)^{-1-\frac{1}{2q}(2-\frac{1}{r})} \quad (t \geq 0).\end{aligned}$$

$$(4) \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| U^r(t, x) - u^r\left(\frac{x}{t}\right) \right| = 0.$$

Because the proofs of Lemmas 2.1 and 2.2 are well-known (see [6, 7, 14, 22–26, 32, 34, 41, 42], and so on), we state here only. We note that we need the smoothness of the convective flux f , that is, $f \in C^4(\mathbb{R})$. This is because the *a priori* estimates, Proposition 4.3 in Section 4, cannot be established without using the estimates in (3) in Lemma 2.1, in particular,

$$\|\partial_x^3 U^r(t)\|_{L^2} \leq C_q (1+t)^{-1-\frac{3}{4q}} \quad (\exists C_q > 0)$$

with $q > 1/2$ for $t \geq 0$.

3. Reformulation of the rarefaction problem.

In this section, we reformulate our problem (1.1) in terms of the deviation from the asymptotic state. Now letting

$$u(t, x) = U^r(t, x) + \phi(t, x), \tag{3.1}$$

we reformulate the problem (1.1) in terms of the deviation ϕ from U^r as

$$\begin{cases} \partial_t \phi + \partial_x (f(\phi + U^r) - f(U^r)) \\ -\mu \partial_x^2 \phi + \delta \partial_x^3 \phi = F(U^r) \quad (t > 0, x \in \mathbb{R}), \\ \phi(0, x) = \phi_0(x) := u_0(x) - U^r(0, x) \rightarrow 0 \quad (x \rightarrow \pm\infty), \end{cases} \quad (3.2)$$

where $F(U^r) := \mu \partial_x^2 U^r - \delta \partial_x^3 U^r$.

Then by noting (4) in Lemma 2. 2, we look for the unique global in time solution ϕ which has the asymptotic behavior

$$\sup_{x \in \mathbb{R}} |\phi(t, x)| \rightarrow 0 \quad (t \rightarrow \infty). \quad (3.3)$$

Here we note that $\phi_0 \in H^1$ by the assumptions on u_0 , and Lemma 2. 1. Then the corresponding theorems for ϕ to Theorems 1. 3 and 1.4 we should prove are as follows.

Theorem 3.1 (*Global Existence I*). *Assume the far field states u_{\pm} satisfy $u_- < u_+$, the convective flux $f \in C^4(\mathbb{R})$ satisfy (1.2) and (1.5). Further assume the initial data satisfy $\phi_0 \in H^1$. Then the Cauchy problem (3.2) has a unique global in time solution ϕ satisfying*

$$\begin{cases} \phi \in C^0([0, \infty); H^1), \\ \partial_x \phi \in L^2(0, \infty; H^1), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |\phi(t, x)| = 0.$$

Theorem 3.2 (*Global Existence II*). *Assume the far field states u_{\pm} satisfy $u_- < u_+$, the convective flux $f \in C^4(\mathbb{R})$ satisfy (1.2) and (1.6). Further assume the initial data satisfy $\phi_0 \in H^1$. Then the same result as in Theorem 3.1 holds true.*

In order to accomplish the proof of Theorems 3.1 and 3.2, we prepare the local existence precisely, and we formulate the problem (3.3) at general initial time $\tau \geq 0$:

$$\begin{cases} \partial_t \phi + \partial_x (f(\phi + U^r) - f(U^r)) \\ -\mu \partial_x^2 \phi + \delta \partial_x^3 \phi = F(U^r) \quad (t > \tau, x \in \mathbb{R}), \\ \phi(\tau, x) = \phi_{\tau}(x) := u_{\tau}(x) - U^r(\tau, x) \rightarrow 0 \quad (x \rightarrow \pm\infty). \end{cases} \quad (3.3)$$

Theorem 3.3 (*Local Existence*). *For any $M > 0$, there exists a positive constant $t_0 = t_0(M)$ not depending on τ such that if $\phi_{\tau} \in H^1$ and $\|\phi_{\tau}\|_{H^1} \leq M$, then the Cauchy problem (3.5) has a unique solution ϕ on the time interval $[\tau, \tau + t_0(M)]$ satisfying*

$$\begin{cases} \phi \in C^0([\tau, \tau + t_0]; H^1), \\ \partial_x \phi \in L^2(\tau, \tau + t_0; H^1), \\ \sup_{t \in [\tau, \tau + t_0]} \|\phi(t)\|_{H^1} \leq 2M. \end{cases}$$

Because the proof of Theorem 3.3 is given by the standard iterative approximation method, we omit the details here (cf. [6,7,32,34,41]). The a priori estimates we establish in Sections 4 and 5 are the following.

Theorem 3.4 (*A Priori Estimates II*). *Under the same assumptions as in Theorem 3.1, for any initial data $\phi_0 \in H^1$, there exists a positive constant C_{ϕ_0} such that if the Cauchy problem (3.3) has a solution ϕ on the time interval $[0, T]$ satisfying*

$$\begin{cases} \phi \in C^0([0, T]; H^1), \\ \partial_x \phi \in L^2(0, T; H^1), \end{cases}$$

for some constant $T > 0$, then it holds that

$$\begin{aligned} & \|\phi(t)\|_{H^1}^2 + \int_0^t \int_{-\infty}^{\infty} \int_0^{\phi} (f'(\eta + U^r) - f'(U^r)) \, d\eta \, \partial_x U^r \, dx \, d\tau \\ & + \int_0^t \left(\sup_{x \in \mathbb{R}} |\phi(\tau, x)| \right)^4 \, d\tau + \int_0^t \|\partial_x \phi(\tau)\|_{H^1}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]). \end{aligned} \quad (3.4)$$

Theorem 3.5 (*A Priori Estimates II*). *Under the same assumptions as in Theorem 3.2, for any initial data $\phi_0 \in H^1$, there exists a positive constant C_{ϕ_0} such that if the Cauchy problem (3.3) has a solution ϕ on the time interval $[0, T]$ satisfying*

$$\begin{cases} \phi \in C^0([0, T]; H^1), \\ \partial_x \phi \in L^2(0, T; H^1), \end{cases}$$

for some constant $T > 0$, then it holds that

$$\begin{aligned} & \|\phi(t)\|_{H^1}^2 + \int_0^t \int_{-\infty}^{\infty} \int_0^{\phi} (f'(\eta + U^r) - f'(U^r)) \, d\eta \, \partial_x U^r \, dx \, d\tau \\ & + \int_0^t \left(\sup_{x \in \mathbb{R}} |\phi(\tau, x)| \right)^4 \, d\tau + \int_0^t \int_{-\infty}^{\infty} f''(\phi + U^r) |\partial_x \phi|^2 \partial_x U^r \, dx \, d\tau \\ & + \int_0^t \|\partial_x \phi(\tau)\|_{H^1}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]). \end{aligned} \quad (3.5)$$

Theorems 3.4 and 3.5 which we establish in Sections 5 and 6 are corresponded to Theorems 3.1 and 3.2. Combining the local existence Theorem 3.3 together with the each *a priori* estimates, Theorems 3.4 and 3.5, we can obtain global existence Theorems 3.1 and 3.2, respectively, and the following uniform energy estimates

$$\sup_{t \geq 0} \|\phi(t)\|_{H^1} + \int_0^{\infty} \|\partial_x \phi(t)\|_{H^1}^2 \, dt < \infty. \quad (3.6)$$

From (3.6) we easily have

$$\int_0^{\infty} \left| \frac{d}{dt} \|\partial_x \phi(t)\|_{L^2}^2 \right| \, dt < \infty. \quad (3.7)$$

Therefore, by using (3.6) and (3.7), we have the L^2 -stability as follows.

$$\|\partial_x \phi(t)\|_{L^2} = \|\partial_x u(t) - \partial_x U^r(t)\|_{L^2} \rightarrow 0 \quad (t \rightarrow \infty). \quad (3.8)$$

By the Sobolev inequality, we have the asymptotic behavior, that is,

$$\sup_{x \in \mathbb{R}} |\phi(t, x)| \leq \sqrt{2} \|\phi(t)\|_{L^2}^{\frac{1}{2}} \|\partial_x \phi(t)\|_{L^2}^{\frac{1}{2}} \rightarrow 0 \quad (t \rightarrow \infty). \quad (3.9)$$

4. Remarks on the uniform estimates.

It is worthwhile to mention the uniform estimates of the solution ϕ to (3.2). From the first term on the left-hand side of (3.4) in the *a priori* estimates in Theorem 3.4, by using the Sobolev inequality, we obtain the following uniform boundedness of ϕ as

$$\sup_{t \in [0, T], x \in \mathbb{R}} |\phi(t, x)| \leq C_{\phi_0} \quad (4.1)$$

(cf. [16, 18, 22, 28]). By (4.1), we immediately have that

$$\begin{aligned} & \int_0^t \int_{-\infty}^{\infty} \int_0^{\phi} (f'(\eta + U^r) - f'(U^r)) \, d\eta \, \partial_x U^r \, dx \, d\tau \\ &= \int_0^t \int_{-\infty}^{\infty} (f(\phi + U^r) - f(U^r) - f'(U^r) \phi) \, \partial_x U^r \, dx \, d\tau \\ &\geq C_{\phi_0}^{-1} \int_0^t \|\sqrt{\partial_x U^r} \phi(\tau)\|_{L^2}^2 \, d\tau \quad (t \in [0, T]). \end{aligned}$$

Also from the first and fourth terms on the left-hand side of (3.4), we have by using

$$\left(\sup_{x \in \mathbb{R}} |\partial_x \phi(t, x)| \right)^2 \leq \min \{ \|\partial_x \phi(\tau)\|_{H^1}^2, C_{\phi_0} \|\partial_x \phi(\tau)\|_{L^2}^2 \} \quad (t \in [0, T]),$$

which is given by the Sobolev inequality and the integration by parts, that

$$\int_0^t \left(\sup_{x \in \mathbb{R}} |\partial_x \phi(\tau, x)| \right)^2 \, d\tau + \int_0^t \left(\sup_{x \in \mathbb{R}} |\partial_x \phi(\tau, x)| \right)^4 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]). \quad (4.2)$$

We finally remark by

$$\begin{aligned} \sup_{x \in \mathbb{R}} |\phi(t, x)| &\leq \sqrt{2} \|\phi(t)\|_{L^2}^{\frac{1}{2}} \|\partial_x \phi(t)\|_{L^2}^{\frac{1}{2}} \\ &\leq \sqrt{2} \|\phi(t)\|_{L^2}^{\frac{3}{4}} \|\partial_x^2 \phi(t)\|_{L^2}^{\frac{1}{4}} \quad (t \in [0, T]) \end{aligned}$$

that

$$\int_0^t \left(\sup_{x \in \mathbb{R}} |\phi(\tau, x)| \right)^8 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]). \quad (4.3)$$

5. *A priori* estimates I.

In this section, we show the following *a priori* estimate for ϕ in Theorem 3.4. Noting $f''(s) > 0$ for $\forall s \in \mathbb{R}$, (1.2), multiplying the equation in (3.2) by ϕ and integrating the resultant formula with respect to x and t , we obtain the *a priori* estimates for ϕ as follows (for the proof, see [6, 7, 32, 34] and so on).

Proposition 5.1. *Suppose that the convective flux $f \in C^4(\mathbb{R})$ satisfies (1.2). Then, there exists a positive constant C_{ϕ_0} such that*

$$\begin{aligned} & \|\phi(t)\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} \int_0^{\phi} (f'(\eta + U^r) - f'(U^r)) \, d\eta \, \partial_x U^r \, dx \, d\tau \\ & + \int_0^t \left\| (\sqrt{\partial_x U^r} \phi)(\tau) \right\|_{L^2(|\phi| \leq 1)}^2 \, d\tau + \int_0^t \|\partial_x \phi(\tau)\|_{L^2}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]). \end{aligned}$$

From Proposition 4.1, by using the Sobolev inequality, we immediately have the next lemma.

Lemma 5.2. *Suppose that the convective flux $f \in C^4(\mathbb{R})$ satisfies (1.2). Then, there exists a positive constant C_{ϕ_0} such that*

$$\int_0^t \left(\sup_{x \in \mathbb{R}} |\phi(\tau, x)| \right)^4 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]),$$

for some positive constant C_{ϕ_0} .

Next, under the assumption

$$|f'(u)| \leq O(1) (1 + |u|^p) \quad (p \geq 0), \quad (5.1)$$

we show the a priori estimate for $\partial_x \phi$ as follows.

Proposition 5.3. *For $0 \leq p \leq 2$, there exists a positive constant C_{ϕ_0} such that*

$$\|\partial_x \phi(t)\|_{L^2}^2 + \int_0^t \|\partial_x^2 \phi(\tau)\|_{L^2}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]).$$

Proof of Proposition 5.3. Because the case $0 \leq p \leq 1$ is easy, we assume $1 < p$ in what follows. Multiplying the equation in (3.2) by $-\partial_x^2 \phi$, and integrating the resultant formula with respect to x , we have, after integration by parts, that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x \phi(t)\|_{L^2}^2 + \mu \|\partial_x^2 \phi(t)\|_{L^2}^2 \\ & = \int_{-\infty}^{\infty} \partial_x^2 \phi \, \partial_x (f(\phi + U^r) - f(U^r)) \, dx - \int_{-\infty}^{\infty} \partial_x^2 \phi \, F(U^r) \, dx. \end{aligned} \quad (5.2)$$

First, by using the Young inequality, the second term on the right-hand side of (5.2) is easily estimated as follows.

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \partial_x^2 \phi \, F(U^r) \, dx \right| & \leq \max\{\mu, \delta\} \int_{-\infty}^{\infty} |\partial_x^2 \phi| (|\partial_x^2 U^r| + |\partial_x^3 U^r|) \, dx \\ & \leq \frac{\mu}{4} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu, \delta} \|\partial_x^2 U^r(t)\|_{H^1}^2. \end{aligned} \quad (5.3)$$

We estimate the first term on the right-hand side as follows.

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} \partial_x^2 \phi \, \partial_x (f(\phi + U^r) - f(U^r)) \, dx \right| \\ & \leq \left| \int_{-\infty}^{\infty} \partial_x \phi \, \partial_x^2 \phi \, f'(\phi + U^r) \, dx \right| + \left| \int_{-\infty}^{\infty} \partial_x U^r \, \partial_x^2 \phi (f'(\phi + U^r) - f'(U^r)) \, dx \right|. \end{aligned} \quad (5.4)$$

Noting the assumption (5.1), using the Young inequality and the Sobolev inequality, each terms on the right-hand side of (5.4) are estimated as follows.

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} \partial_x \phi \partial_x^2 \phi f'(\phi + U^r) dx \right| \\ & \leq \frac{\mu}{4} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu,p} \int_{-\infty}^{\infty} |\partial_x \phi|^2 (1 + |\phi|^{2p}) dx \end{aligned} \quad (5.5)$$

$$\begin{aligned} & \leq \frac{\mu}{4} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu,p} \|\partial_x \phi(t)\|_{L^2}^2 + C_{\mu,p} \left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^{2p} \|\partial_x \phi(t)\|_{L^2}^2, \\ & \left| \int_{-\infty}^{\infty} \partial_x U^r \partial_x^2 \phi (f'(\phi + U^r) - f'(U^r)) dx \right| \\ & \leq \int_{-\infty}^{\infty} |\partial_x U^r| |\partial_x^2 \phi| |\phi| \left| \frac{f'(\phi + U^r) - f'(U^r)}{\phi} \right| dx \\ & \leq C_p \int_{-\infty}^{\infty} |\partial_x U^r| |\phi| |\partial_x^2 \phi| dx + C_p \int_{-\infty}^{\infty} |\partial_x U^r| |\phi|^p |\partial_x^2 \phi| dx. \end{aligned} \quad (5.6)$$

Each terms on the right-hand side of (5.6) are estimated as follows.

$$\begin{aligned} & C_p \int_{-\infty}^{\infty} |\partial_x U^r| |\phi| |\partial_x^2 \phi| dx \\ & \leq \frac{\mu}{8} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu,p} \int_{-\infty}^{\infty} |\partial_x U^r|^2 |\phi|^2 dx \\ & \leq \frac{\mu}{8} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu,p} \left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^2 \|\partial_x U^r(t)\|_{L^2}^2 \\ & \leq \frac{\mu}{8} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu,p} \left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^4 + C_{\mu,p} \|\partial_x U^r(t)\|_{L^2}^4, \end{aligned} \quad (5.7)$$

$$\begin{aligned} & C_p \int_{-\infty}^{\infty} |\partial_x U^r| |\phi|^p |\partial_x^2 \phi| dx \\ & \leq \frac{\mu}{8} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu,p} \int_{-\infty}^{\infty} |\partial_x U^r|^2 |\phi|^{2p} dx \\ & \leq \frac{\mu}{8} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu,p} \left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^{2p-2} \|\phi(t)\|_{L^2}^2 \|\partial_x U^r(t)\|_{L^\infty}^2. \end{aligned} \quad (5.8)$$

If $1 < p \leq 2$, substituting (5.3)-(5.8) into (5.2), integrating the resultant formula with respect to t , noting Proposition 5.1, and Lemmas 2.2 and 4.2, and further using the Gronwall inequality, we obtain the desired estimate, Proposition 5.3.

Thus, the proof of Proposition 5.3 is completed.

6. *A priori* estimates II.

In this section, under the assumption

$$|f''(u)| \leq O(1) (1 + |u|^q) \quad (q \geq 0), \quad (6.1)$$

we show the following *a priori* estimate for ϕ in Theorem 3.5. First, it should be noted that the same statements as those in Proposition 5.1 and Lemma 5.2 in Section 5 hold true. This is because the proofs of Proposition 5.1

and Lemma 5.2 are not necessarily further conditions for f , such as (5.1) and (6.1). Therefore, we only show the next proposition.

Proposition 6.1. *For $0 \leq q \leq 1$, there exists a positive constant C_{ϕ_0} such that*

$$\begin{aligned} & \| \partial_x \phi(t) \|_{L^2}^2 + \int_0^t \| \partial_x^2 \phi(\tau) \|_{L^2}^2 d\tau \\ & + \int_0^t \int_{-\infty}^{\infty} f''(\phi + U^r) | \partial_x \phi |^2 \partial_x U^r dx d\tau \leq C_{\phi_0} \quad (t \in [0, T]). \end{aligned}$$

Proof of Proposition 6.1. By using the integration by parts, (5.2) further becomes

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \| \partial_x \phi(t) \|_{L^2}^2 + \mu \| \partial_x^2 \phi(t) \|_{L^2}^2 + \frac{3}{2} \int_{-\infty}^{\infty} f''(\phi + U^r) \partial_x U^r | \partial_x \phi |^2 dx \\ & = - \int_{-\infty}^{\infty} | \partial_x U^r |^2 \partial_x \phi (f''(\phi + U^r) - f''(U^r)) dx - \int_{-\infty}^{\infty} \partial_x^2 U^r \partial_x \phi (f'(\phi + U^r) - f'(U^r)) dx \quad (6.2) \\ & - \frac{1}{2} \int_{-\infty}^{\infty} (\partial_x \phi)^3 f''(\phi + U^r) dx - \int_{-\infty}^{\infty} \partial_x^2 \phi F(U^r) dx. \end{aligned}$$

The fourth term on the right-hand side of (6.2) is estimated as (5.3). By (6.1), the first and second terms on the right-hand side of (6.2) are estimated as follows.

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} | \partial_x U^r |^2 \partial_x \phi (f''(\phi + U^r) - f''(U^r)) dx \right| \\ & \leq C_q \int_{-\infty}^{\infty} | \partial_x U^r |^2 | \partial_x \phi | dx + C_q \int_{-\infty}^{\infty} | \phi |^q | \partial_x U^r |^2 | \partial_x \phi | dx \quad (6.3) \\ & \leq C_q \left(1 + \left(\sup_{x \in \mathbb{R}} | \phi(t, x) | \right)^q \right) (\| \partial_x U^r(t) \|_{L^4}^4 + \| \partial_x \phi(t) \|_{L^2}^2), \end{aligned}$$

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} \partial_x^2 U^r \partial_x \phi (f'(\phi + U^r) - f'(U^r)) dx \right| \\ & \leq C_q \int_{-\infty}^{\infty} | \phi | | \partial_x^2 U^r | | \partial_x \phi | dx + C_q \int_{-\infty}^{\infty} | \phi |^{q+1} | \partial_x^2 U^r | | \partial_x \phi | dx \quad (6.4) \\ & \leq C_q \left(\sup_{x \in \mathbb{R}} | \phi(t, x) | + \left(\sup_{x \in \mathbb{R}} | \phi(t, x) | \right)^{q+1} \right) (\| \partial_x^2 U^r(t) \|_{L^2}^2 + \| \partial_x \phi(t) \|_{L^2}^2), \end{aligned}$$

We finally estimate the third term on the right-hand side of (6.2) as follows.

$$\left| \int_{-\infty}^{\infty} (\partial_x \phi)^3 f''(\phi + U^r) dx \right| \leq C_q \| \partial_x \phi(t) \|_{L^3}^3 + C_q \int_{-\infty}^{\infty} | \phi |^q | \partial_x \phi |^3 dx, \quad (6.5)$$

$$\begin{aligned} C_q \| \partial_x \phi(t) \|_{L^3}^3 & \leq C_q \| \partial_x \phi(t) \|_{L^\infty} \| \partial_x \phi(t) \|_{L^2}^2 \\ & \leq C_q \| \partial_x \phi(t) \|_{L^2}^{\frac{5}{2}} \| \partial_x^2 \phi(t) \|_{L^2}^{\frac{1}{2}} \\ & \leq \frac{\mu}{4} \| \partial_x^2 \phi(t) \|_{L^2}^2 + C_{\mu, q} \| \partial_x \phi(t) \|_{L^2}^{\frac{10}{3}} \quad (6.6) \\ & \leq \frac{\mu}{4} \| \partial_x^2 \phi(t) \|_{L^2}^2 + C_{\mu, q} \| \partial_x \phi(t) \|_{L^2}^2 (1 + \| \partial_x \phi(t) \|_{L^2}^2), \end{aligned}$$

$$\begin{aligned}
& C_q \int_{-\infty}^{\infty} |\phi|^q |\partial_x \phi|^3 dx \\
& \leq C_q \left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^q \|\partial_x \phi(t)\|_{L^\infty} \|\partial_x \phi(t)\|_{L^2}^2 \\
& \leq C_q \left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^q \|\partial_x \phi(t)\|_{L^2}^{\frac{5}{2}} \|\partial_x^2 \phi(t)\|_{L^2}^{\frac{1}{2}} \\
& \leq \frac{\mu}{4} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu, q} \left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^{\frac{4q}{3}} \|\partial_x \phi(t)\|_{L^2}^{\frac{10}{3}} \\
& \leq \frac{\mu}{4} \|\partial_x^2 \phi(t)\|_{L^2}^2 + C_{\mu, q} \left(\left(\sup_{x \in \mathbb{R}} |\phi(t, x)| \right)^{4q} + \|\partial_x \phi(t)\|_{L^2}^2 \right) \|\partial_x \phi(t)\|_{L^2}^2.
\end{aligned} \tag{6.7}$$

If $0 \leq q \leq 1$, substituting (5.3) and (6.3)-(6.7) into (6.2), integrating the resultant formula with respect to t , noting Proposition 5.1, and Lemmas 2.2 and 4.2, and further using the Gronwall inequality, we obtain the desired estimate, Proposition 6.1.

Thus, the proof of Proposition 6.1 is completed.

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