初等的な線形函数方程式と関連する話題

Elementary Linear Functional Equations and Related Topics

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Abstract. In this paper, we suggest teaching materials of elementary mathematical analysis, in particular several types of one-dimensional scalar linear functional equations and related topics. We particularly treat linear ordinary differential equations, linear integral equations and linear integro-differential equations. We also explain how the solutions to equations can be obtained only by using some simple technique.

要旨：本論文では、初等的な数学解析に関する教材を提案する。特に１次元単独の線形函数方程式及び、関連する話題について考える。特に、線形常微分方程式、線形積分方程式、線形積分微分方程式を扱う。更に、これら方程式の解が幾つかのシンプルな解法によって求積出来ることも紹介する。

1. Introduction and the Laplace transform method.

We consider several types of scalar one-dimensional linear functional equations (in particular, linear ordinary differential equations, linear integral equations and linear integro-differential equations) and related topics for post-secondary education for mathematical analysis, which can be solved only by using some simple technique such as differential and integral calculus.

The differential, integral and integro-differential equations are the equations which involve unknown functions and their derivatives, unknown functions and their integrals, and unknown functions, their derivatives and integrals (see [1-11] and so on). For example, the first order integro-differential equations (scalar one-dimensional case) have the form

$$x'(t) + \int_{\Omega} f\big(\,t, \tau, x(\tau)\,\big)\,\mathrm{d}\tau = g\big(\,t, x(t)\,\big), \tag{1.1}$$

where f, g are given funcions, x is the unknown function and $\Omega \subset \mathbb{R}$ is a bounded or unbounded domain.

When f, g and Ω are given as

$$\Omega = [0,\,t], \quad f\big(\,t, \tau, x(\tau)\,\big) = \mathrm{e}^{t-\tau}\,x(\tau), \quad g\big(\,t, x(t)\,\big) = \sin t - x(t),$$

then (1.1) becomes

$$x'(t) + x(t) + \int_0^t \mathrm{e}^{t-\tau}\,x(\tau)\,\mathrm{d}\tau = \sin t, \tag{1.2}$$

which is given in Maruo [5] (see also [4], [10]) and an example of the first-order inhomogeneous linear integro-differential equation. We consider (1.2) with a condition $x(0) = 0$, that is, the following problem

$$\begin{cases} x'(t)+x(t)+\displaystyle\int_0^t \mathrm{e}^{t-\tau}\,x(\tau)\,\mathrm{d}\tau=\sin t, \\ x(0)=0. \end{cases} \tag{1.3}$$

In [5] (and [10]), (1.3) is solved by using the Laplace transform method. In fact, applying the Laplace transform on both side of the equation in (1.3), we have

$$\mathcal{L}\left[x'(t)+x(t)+\int_0^t \mathrm{e}^{t-\tau}\,x(\tau)\,\mathrm{d}\tau\right](s)=\mathcal{L}\left[\sin t\right](s). \tag{1.4}$$

Here, $\mathcal{L}[f]=\mathcal{L}[f](s)$ is the Laplace transform of $f=f(t)$ and defined as follows (see [10] and so on).

$$\mathcal{L}[f](s)=\overset{t\to s}{\mathcal{L}}[f](s):=\int_0^\infty \mathrm{e}^{-st}\,f(t)\,\mathrm{d}t,$$

where $s=\mathrm{Re}\{s\}+\sqrt{-1}\,\mathrm{Im}\{s\}\in\mathbb{C}$ is a frequency parameter and f is a function defined on $[0,\infty)$ satisfying

$$|f(t)|\le K\,\mathrm{e}^{\alpha t}\quad\big(\exists K>0,\ \mathrm{Re}\{s\}>\exists\alpha>0\big).$$

Noting the linearlity of the Laplace transform and the condition $x(0)=x(+0)=0$, and using

$$\mathcal{L}\left[\mathrm{e}^{\pm\alpha t}\right](s)=\frac{1}{s\mp\alpha}\quad\big(\alpha\in\mathbb{R},\ \mathrm{Re}\{s\}>\pm\alpha\big),\quad \mathcal{L}\left[\sin\omega t\right](s)=\frac{\omega}{s^2+\omega^2}\quad\big(\omega\in\mathbb{R},\ \mathrm{Re}\{s\}>0\big),$$
$$\mathcal{L}\left[x'\right](s)=s\,\mathcal{L}\left[x\right](s)-x(+0)=s\,\mathcal{L}\left[x\right](s),$$
$$\mathcal{L}\left[\int_0^t \mathrm{e}^{t-\tau}\,x(\tau)\,\mathrm{d}\tau\right](s)=\mathcal{L}\left[\mathrm{e}^t*x(t)\right](s)=\mathcal{L}\left[\mathrm{e}^t\right](s)\,\mathcal{L}\left[x(t)\right](s)=\frac{1}{s-1}\,\mathcal{L}\left[x(t)\right](s),$$

then, (1.4) becomes

$$\mathcal{L}\left[x\right](s)=\frac{s-1}{s\left(s^2+1\right)}=-\frac{1}{s^2}+\frac{1}{s}-\frac{s}{s^2+1}+\frac{1}{s^2+1}. \tag{1.5}$$

Therefore, applying the inverse Laplace transform on both side of (1.5), and noting the Fourier inversion theorem and the linearlity of the inverse Laplace transform, we have

$$x(t)=\mathcal{L}^{-1}\left[\mathcal{L}\left[x\right]\right](t)=-\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t)+\mathcal{L}^{-1}\left[\frac{1}{s}\right](t)-\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right](t)+\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t). \tag{1.6}$$

Here, $\mathcal{L}^{-1}[g]=\mathcal{L}^{-1}[g](t)$ is the inverse Laplace transform of $g=g(s)$ and defined as follows (see [10] and so on).

$$\mathcal{L}^{-1}[g](t)=\overset{s\to t}{\mathcal{L}^{-1}}[g](t):=\frac{1}{2\pi\sqrt{-1}}\lim_{R\to\infty}\int_{\mathrm{Re}\{s\}-\sqrt{-1}\,R}^{\mathrm{Re}\{s\}+\sqrt{-1}\,R}\mathrm{e}^{st}\,g(s)\,\mathrm{d}s.$$

By using

$$\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right](s)=t^n\quad\big(n\in\mathbb{N}\cup\{0\},\ \mathrm{Re}\{s\}>0\big),$$
$$\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right](s)=\sin\omega t\quad\big(\omega\in\mathbb{R},\ \mathrm{Re}\{s\}>0\big),\quad \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right](s)=\cos\omega t\quad\big(\omega\in\mathbb{R},\ \mathrm{Re}\{s\}>0\big),$$

we thus obtain the desired solutin to (1.3) as follows.

$$x(t)=-t+1-\cos t+\sin t. \tag{1.7}$$

However, we can solve (1.3) more simple way. In fact, differentiating the equation in (1.3), we have

$$
\begin{aligned}
&x''(t)+x'(t)+\mathrm{e}^{t-\tau}\,x(\tau)\big|_{t=\tau}+\int_0^t \frac{\partial}{\partial t}\,\{\,\mathrm{e}^{t-\tau}\,x(\tau)\,\}\,\mathrm{d}\tau=\cos t\\
&\therefore x''(t)+x'(t)+x(t)+\int_0^t \mathrm{e}^{t-\tau}\,x(\tau)\,\mathrm{d}\tau=\cos t.
\end{aligned}
\tag{1.8}
$$

Further differentiating (1.8), we have

$$
x'''(t)+x''(t)+x'(t)+x(t)+\int_0^t \mathrm{e}^{t-\tau}\,x(\tau)\,\mathrm{d}\tau=-\sin t. \tag{1.9}
$$

Substituting $t=0$ into (1.3) and (1.8), we have $x'(0)=0$ and $x''(0)=1$, respectively. Also substituting (1.9) into (1.3), we have the problem for the third order inhomogeneous linear ordinary differential equation corresponded to (1.3) as follows.

$$
\begin{cases}
(\,x'+x\,)''=-2\,\sin t,\\
x(0)=x'(0)=0,\quad x''(0)=1.
\end{cases}
\tag{1.10}
$$

Integrating twice the equation in (1.10), we have

$$
x'+x=2\,\sin t+C_1\,t+C_2. \tag{1.11}
$$

Noting $C_1=-1$ and $C_2=0$ from $(\,x'+x\,)'\,(0)=0$, we arrive at a problem for first order inhomogeneous linear ordinary differential equation corresponded to (1.3) or (1.11) as follows.

$$
\begin{cases}
x'+x=2\,\sin t-t,\\
x(0)=x'(0)=0.
\end{cases}
\tag{1.12}
$$

Multiplying the equation in (1.12) by e^t, integrating it and dividing the resultant formula by e^t, we obtain

$$
x(t)=\mathrm{e}^{-t}\int \mathrm{e}^t\,(\,2\,\sin t-t\,)\,\mathrm{d}t+C_3\,\mathrm{e}^{-t}=\sin t-\cos t-t+1+C_3\,\mathrm{e}^{-t}. \tag{1.13}
$$

Here, we note $C_3=0$ from $x(0)=0$. Therefore, we have (1.7).

In the following sections, without using the Laplace transform method, similarly in Yoshida [11], we give many examples of teaching materials for linear functional equations by using simple technique such as the process (1.8)-(1.13).

This paper is organized as follows. In Section 2, we give several examples of teaching materials for linear ordinary differential equations. For linear integral equations such as the Volterra integral equations and linear integro-differential equations, we also give many examples in Sections 3 and 4.

2. Examples for linear ordinary differential equations.

In this section, we give several examples of teaching materials for linear ordinary differential equations only by using simple technique.

Example 2.1. Solve the following problem.

$$
\begin{cases}
t\,x''-(1+t)\,x'+2\,x=t-1,\\
x(0)=0,\quad x'(1)=3.
\end{cases}
\tag{2.1}
$$

Solution: We should note the equation in (2.1) is a second-order linear ordinary differential equation with variable coefficients. Therefore, in general, the method with differential operator (see Theorem 4.1 in Section 4) cannot be applied to it. Noting $(t^n)' = n\,t^{n-1}$ $(n \in \mathbb{N})$ and the representation of the inhomogeneous term $t-1$, we look for the solution which has the form

$$x(t) = a\,t^2 + b\,t + c, \tag{2.2}$$

and determine the coefficients a, b and c suitably in order to satisfy (2.1). Differentiating (2.2), we have

$$x'(t) = 2\,a\,t + b, \quad x''(t) = 2\,a. \tag{2.3}$$

Substituting (2.2) and (2.3) into the equation in (2.1), we have

$$b\,t + 2\,c - b = t - 1 \quad \therefore (b, c) = (1, 0). \tag{2.4}$$

Substituting (2.4) into (2.2), we have

$$x(t) = a\,t^2 + b\,t + c, \quad x'(t) = 2\,a\,t, \tag{2.5}$$

which satisfy $x(0) = 0$. Further from $x'(1) = 3$, we also have $a = 1$. Substituting this and (2.4) into (2.2), we obtain the desired solution to (2.1) as follows.

$$x(t) = t^2 + t. \tag{2.6}$$

Remark 2.2. If we use the Laplace transform method to (2.1), we can obtain after some computation that

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}[\,x''\,](s) - \left(1 - \frac{\mathrm{d}}{\mathrm{d}s}\right)\mathcal{L}[\,x'\,](s) + 2\,\mathcal{L}[\,x\,](s) = \mathcal{L}[\,t-1\,](s) \\ &\therefore s\,\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}[\,x\,](s) + 3\,\mathcal{L}[\,x\,](s) = \frac{1}{s^2}. \end{aligned} \tag{2.7}$$

The solution to the first order linear ordinary differential equation (2.7) is

$$\mathcal{L}[\,x\,](s) = \frac{1}{s^2} + \frac{C}{s^3} = \mathcal{L}\left[\,\frac{1}{t} + \frac{C}{2}\,t^2\,\right](s). \tag{2.8}$$

We get $C = 2$ from $x'(1) = 3$ and therefore obtain (2.6) from (2.8).

Example 2.3. Solve the following problem.

$$\begin{cases} t\,x''' + (1 + t^2)\,x'' + t\,x' + t\,x = t^2 + 1, \\ x(0) = -1, \quad x'(0) = 1, \quad x''(0) = 0. \end{cases} \tag{2.9}$$

Solution: Noting the representation of the inhomogeneous term $t^2 + 1$, we look for the solution which has the form

$$x(t) = a\,t + b, \tag{2.10}$$

and determine the coefficients a and b suitably in order to satisfy (2.9). Differentiating (2.2), we have

$$x'(t) = a, \quad x''(t) = x'''(t) = 0. \tag{2.11}$$

Substituting (2.10) and (2.11) into the equation in (2.9), we have

$$a\,t^2 + (a+b)\,t = t^2 + 1 \qquad \therefore (a,b) = (1,-1). \tag{2.12}$$

Substituting (2.12) into (2.10), we have

$$x(t) = t - 1, \tag{2.13}$$

which satisfies $x(0) = -1$, $x'(0) = 1$ and $x''(0) = 0$. Therefore, (2.13) is the desired solution to (2.9).

Example 2.4. Solve the following problem.

$$\begin{cases} x''' - 8\,t\,x'' - 4\,x' - 64\,t^3\,x = -128\,t^3\,\mathrm{e}^{2t^2}, \\ x(0) = 1, \quad x'(0) = 0, \quad x''(0) = 1. \end{cases} \tag{2.14}$$

Solution: We first note that for $\alpha \in \mathbb{R}$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\alpha t^2} = 2\,\alpha\,t\,\mathrm{e}^{\alpha t^2}, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathrm{e}^{\alpha t^2} = 2\,\alpha\,\mathrm{e}^{\alpha t^2} + \big(2\,\alpha\,t\big)^2\,\mathrm{e}^{\alpha t^2}, \\ &\frac{\mathrm{d}^3}{\mathrm{d}t^3}\,\mathrm{e}^{\alpha t^2} = 12\,\alpha^2\,t\,\mathrm{e}^{\alpha t^2} + \big(2\,\alpha\,t\big)^3\,\mathrm{e}^{\alpha t^2}. \end{aligned} \tag{2.15}$$

Also noting the representation of the inhomogeneous term $-128\,t^3\,\mathrm{e}^{2t^2}$, we look for the solution which has the form

$$x(t) = C\,\mathrm{e}^{2t^2}, \tag{2.16}$$

and determine the coefficient C suitably in order to satisfy (2.14). Putting $\alpha = 2$ to (2.15) or differentiating (2.16), we have

$$x'(t) = 4\,C\,t\,\mathrm{e}^{2t^2}, \quad x''(t) = 4\,C\,\mathrm{e}^{2t^2} + 16\,C\,t^2\,\mathrm{e}^{2t^2}, \quad x'''(t) = 48\,C\,t\,\mathrm{e}^{2t^2} + 64\,C\,t^3\,\mathrm{e}^{2t^2}. \tag{2.17}$$

Putting $t = 0$ to (2.16), we get $C = 1$ and

$$x'(t) = 4\,t\,\mathrm{e}^{2t^2}, \quad x''(t) = 4\,\mathrm{e}^{2t^2} + 16\,t^2\,\mathrm{e}^{2t^2}, \quad x'''(t) = 48\,t\,\mathrm{e}^{2t^2} + 64\,t^3\,\mathrm{e}^{2t^2}. \tag{2.18}$$

Thus, we can conclude that

$$x(t) = \mathrm{e}^{2t^2} \tag{2.19}$$

is the solution to (2.14) by substituting (2.18) and (2.19) into (2.14).

Example 2.5. Solve the following problem.

$$\begin{cases} t\,x'' + 2\,x' - (t-2)\,x = 2\,\mathrm{e}^t, \\ x(0) = 0, \quad x'(0) = 1. \end{cases} \tag{2.20}$$

Solution: The equation in (2.20) becomes as follows.

$$t\,\big(\,x'' - x\,\big) + 2\,\big(\,x' + x\,\big) = 2\,\mathrm{e}^t. \tag{2.21}$$

If $\tilde{x} = \tilde{x}(t)$ satisfies the following problem

$$
\begin{cases} \tilde{x}'' - \tilde{x} = 0, \quad \tilde{x}' + \tilde{x} = \mathrm{e}^t, \\ \tilde{x}(0) = 0, \quad \tilde{x}'(0) = 1, \end{cases} \tag{2.22}
$$

then it is clear that $x = \tilde{x}$ is the solution to (2.20). Therefore, we look for the solution to (2.22). We know that

$$
\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sinh t = \cosh t, \quad \frac{\mathrm{d}}{\mathrm{d}t}\cosh t = \sinh t, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\sinh t = \sinh t, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\cosh t = \cosh t, \\ &\cosh t + \sinh t = \mathrm{e}^t. \end{aligned} \tag{2.23}
$$

If we choose

$$
\tilde{x}(t) = \sinh t, \tag{2.24}
$$

then (2.24) satisfies (2.23), $\tilde{x}(0) = 0$ and $\tilde{x}'(0) = 1$, that is, (2.22). Therefore, (2.24) is the desired solution to (2.20).

Example 2.6. Solve the following problem.

$$
\begin{cases} \big(1 + t^2\big)\, x''' + \mathrm{e}^t\, x'' + (1+t)\,\mathrm{e}^{2t}\, x' + \big(\mathrm{e}^t + \mathrm{e}^{2t} + t\,\mathrm{e}^{2t}\big)\, x \\ \quad = (1+t)\,\mathrm{e}^{2t}\,\big(\sin t + \cos t\big) - \big(1 + t^2\big)\cos t, \\ x(0) = 0, \quad x'(0) = 1, \quad x''(0) = 0. \end{cases} \tag{2.25}
$$

Solution: The equation in (2.25) becomes as follows.

$$
\big(1 + t^2\big)\, x''' + \mathrm{e}^t\,\big(x'' + x\big) + (1+t)\,\mathrm{e}^{2t}\,\big(x' + x\big) = -\big(1 + t^2\big)\cos t + (1+t)\,\mathrm{e}^{2t}\,\big(\sin t + \cos t\big). \tag{2.26}
$$

If $\tilde{x} = \tilde{x}(t)$ satisfies the following problem

$$
\begin{cases} \tilde{x}''' + \cos t = 0, \quad \tilde{x}'' + \tilde{x} = 0, \quad \tilde{x}' + \tilde{x} = \sin t + \cos t, \\ \tilde{x}(0) = 0, \quad \tilde{x}'(0) = 1, \quad \tilde{x}''(0) = 0, \end{cases} \tag{2.27}
$$

then it is clear that $x = \tilde{x}$ is the solution to (2.25). We easily see

$$
\tilde{x}(t) = \sin t \tag{2.28}
$$

is the solution to (2.27). Therefore, (2.28) is the desired solution to (2.25).

3. Examples for linear integral equations.

In this section, we give two examples of teaching materials for linear integral equations without using the Laplace transform method. The following Example 3.1 is given and solved by using the Laplace transform method in Kida [4].

Example 3.1. Solve the following Volterra integral equation of the first kind.

$$
\int_0^t \cos(t-\tau)\, x(\tau)\, \mathrm{d}\tau = \sin t + t\, \cos t. \tag{3.1}
$$

Solution: Differentiating (3.1), we have

$$
x(t) - \int_0^t \sin(t-\tau)\, x(\tau)\, \mathrm{d}\tau = 2\cos t - t\, \sin t. \tag{3.2}
$$

Further, differentiating (3.2), we have

$$x'(t) - \int_0^t \cos(t-\tau)\, x(\tau)\, \mathrm{d}\tau = -3 \sin t - t \cos t. \tag{3.3}$$

We get $x(0) = 2$ and $x'(0) = 0$ by putting $t = 0$ to (3.2) and (3.3), respectively, and

$$x' = -2 \sin t \tag{3.4}$$

by substituting (3.1) into (3.3). Then, we arrive at a problem for the first order inhomogeneous linear ordinary differential equation corresponded to (3.1) as follows.

$$\begin{cases} x' = -2 \sin t, \\ x(0) = 2, \quad x'(0) = 0. \end{cases} \tag{3.5}$$

Integrating the equation in (3.5), we obtain the desired solution to (3.5) (and (3.1)) as follows.

$$x(t) = 2 \cos t.$$

Example 3.2. Solve the following Volterra integral equation of the second kind.

$$x(t) = t + \int_0^t \sin(t-\tau)\, x(\tau)\, \mathrm{d}\tau. \tag{3.6}$$

Solution: Differentiating (3.6), we have

$$x'(t) = 1 + \int_0^t \cos(t-\tau)\, x(\tau)\, \mathrm{d}\tau. \tag{3.7}$$

Further, differentiating (3.2), we have

$$x''(t) = x'(t) - \int_0^t \sin(t-\tau)\, x(\tau)\, \mathrm{d}\tau. \tag{3.8}$$

We get $x(0) = 0$ and $x'(0) = 1$ by putting $t = 0$ to (3.6) and (3.7), and $x'' = t$ by substituting (3.1) into (3.3). Thus, we obtain the desired solution to (3.6) as follows.

$$x(t) = \frac{1}{6} t^3 + t.$$

4. Examples for linear integro-differential equations.

In this section, we give some examples of teaching materials for linear integro-differential equations by using the method with differential operator.

To do that, we prepare the next well-known theorems (for the proofs, see [10] and so on).

Theorem 4.1. *Let* $\alpha_k \neq \alpha_l$ $\left(\alpha_k,\, \alpha_l \in \mathbb{C}\right)$ *for any* $k,\, l \in \{1,\, 2,\, \cdots,\, n\}$ $(n \in \mathbb{N})$. *Then, the* n*-th order homogeneous ordinary differential equation*

$$\prod_{k=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t} - \alpha_k \right) x = 0$$

has the following general solution

$$x(t) = \sum_{k=1}^{n} C_k \, \mathrm{e}^{\alpha_k t},$$

where $C_1,\, C_2,\, \cdots,\, C_n$ *are arbtrary constants.*

Theorem 4.2. *The* m*-th order homogeneous ordinary differential equation*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \alpha \right)^m x = 0 \quad \bigl(\alpha \in \mathbb{C},\, m \in \mathbb{N} \bigr)$$

has the following general solution

$$x(t) = \sum_{k=1}^{m} C_k \, t^{k-1} \, \mathrm{e}^{\alpha t},$$

where $C_1,\, C_2,\, \cdots,\, C_m$ *are arbtrary constants.*

Remark 4.3. We note that $\mathrm{e}^{\alpha_1 t},\, \mathrm{e}^{\alpha_2 t},\, \cdots,\, \mathrm{e}^{\alpha_n t}$ in Theorem 4.1 and $1,\, t\,\mathrm{e}^{\alpha t},\, \cdots,\, t^{m-1}\,\mathrm{e}^{\alpha t}$ in Theorem 4.2 are all complex valued functions of a real variable. We also emphasize that a set of the functions

$$\left\{ \mathrm{e}^{\alpha_1 t},\, \mathrm{e}^{\alpha_2 t},\, \cdots,\, \mathrm{e}^{\alpha_n t} \right\}$$

where $\alpha_k \neq \alpha_l \ \bigl(\alpha_k,\, \alpha_l \in \mathbb{C} \bigr)$ for any $k,\, l \in \{1,\, 2,\, \cdots,\, n\}$ in Theorem 4.1 is linearly independent, that is, the following equation

$$\sum_{k=1}^{n} C_k \, \mathrm{e}^{\alpha_k t} = 0$$

has only the trivial solution $C_1 = C_2 = \cdots = C_m = 0$, and a set of the functions

$$\left\{ 1,\, t\,\mathrm{e}^{\alpha t},\, \cdots,\, t^{m-1}\,\mathrm{e}^{\alpha t} \right\}$$

where $\alpha \in \mathbb{C}$ in Theorem 4.2 is also linearly independent.

The following Example 4.3 is given and solved by using the Laplace transform method in Maruo [5].

Example 4.4. Solve the following problem.

$$\begin{cases} x''(t) - x'(t) + x(t) + \displaystyle\int_0^t \mathrm{e}^{t-\tau} \, x(\tau) \, \mathrm{d}\tau = 1, \\ x(0) = A, \quad x'(0) = B, \end{cases} \tag{4.1}$$

where A and B are constants.

Solution: Differentiating (4.1), we have

$$x'''(t) - x''(t) + x'(t) + x(t) + \int_0^t \mathrm{e}^{t-\tau} \, x(\tau) \, \mathrm{d}\tau = 0. \tag{4.2}$$

Substituting (4.1) into (4.2), we have

$$x'''(t) - 2\,x''(t) + 2\,x'(t) + 1 = 0. \tag{4.3}$$

We also get

$$x''(0) = x'(0) - x(0) + 1 = B - A + 1, \quad x'''(0) = 2\,x''(0) - 2\,x'(0) - 1 = -2\,A + 1, \tag{4.4}$$

by putting $t = 0$ to (4.2) and (4.3). Differentiating (4.3), we obtain from (4.1) and (4.4) that

$$\begin{cases} x^{(4)} - 2\,x''' + 2\,x'' = 0, \\ x(0) = A, \quad x'(0) = B, \\ x''(0) = B - A + 1, \quad x'''(0) = -2\,A + 1. \end{cases} \tag{4.5}$$

By Theorems 4.1 and 4.2, we obtain the solution to the problem (4.5) corresponded to (4.1). The equation in (4.5) becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \left\{\frac{\mathrm{d}}{\mathrm{d}t} - \big(1 + \sqrt{-1}\big)\right\} \left\{\frac{\mathrm{d}}{\mathrm{d}t} - \big(1 - \sqrt{-1}\big)\right\} x = 0. \tag{4.6}$$

Using Theorems 4.1 and 4.2 and noting the Euler formula, we obtain the general solution to (4.6) as follows.

$$\begin{aligned} x(t) &= C_0 + C_1\,t + C_2\,\mathrm{e}^{(1+\sqrt{-1})t} + C_3\,\mathrm{e}^{(1-\sqrt{-1})t} \\ &= C_0 + C_1\,t + \big(C_2 + C_3\big)\,\mathrm{e}^t\,\cos t + \sqrt{-1}\,\big(C_2 - C_3\big)\,\mathrm{e}^t\,\sin t. \end{aligned} \tag{4.7}$$

Differentiating (4.7), we obtain

$$\begin{aligned} x'(t) &= C_1 + \big\{C_2 + C_3 + \sqrt{-1}\,\big(C_2 - C_3\big)\big\}\,\mathrm{e}^t\,\cos t \\ &\quad + \big\{\sqrt{-1}\,\big(C_2 - C_3\big) - \big(C_2 + C_3\big)\big\}\,\mathrm{e}^t\,\sin t, \\ x''(t) &= 2\sqrt{-1}\,\big(C_2 - C_3\big)\,\mathrm{e}^t\,\cos t - 2\,\big(C_2 + C_3\big)\,\mathrm{e}^t\,\sin t, \\ x'''(t) &= \big\{2\sqrt{-1}\,\big(C_2 - C_3\big) - 2\,\big(C_2 + C_3\big)\big\}\,\mathrm{e}^t\,\cos t \\ &\quad + \big\{2\sqrt{-1}\,\big(C_2 - C_3\big) + 2\,\big(C_2 + C_3\big)\big\}\,\mathrm{e}^t\,\sin t. \end{aligned} \tag{4.8}$$

Substituting $t = 0$ into (4.8) and using (4.5), we have

$$\begin{aligned} &C_0 + C_2 + C_3 = A, \quad C_1 + C_2 + C_3 + \sqrt{-1}\,\big(C_2 - C_3\big) = B, \\ &2\sqrt{-1}\,\big(C_2 - C_3\big) = B - A + 1, \quad 2\sqrt{-1}\,\big(C_2 - C_3\big) - 2\,\big(C_2 + C_3\big) = -2\,A + 1. \end{aligned} \tag{4.9}$$

Therefore, we get as follows.

$$C_0 = \frac{A - B}{2}, \quad C_1 = -\frac{1}{2}, \quad C_2 + C_3 = \frac{A + B}{2}, \quad \sqrt{-1}\,\big(C_2 - C_3\big) = \frac{B - A + 1}{2}. \tag{4.10}$$

Thus, by substituting (4.10) into (4.7), we obtain the desired solution to (4.1) and (4.5) as follows.

$$x(t) = \frac{1}{2}\,\big(\mathrm{e}^t\,\sin t - t\big) + \frac{A}{2}\,\big\{\mathrm{e}^t\,\big(\cos t - \sin t\big) + 1\big\} + \frac{B}{2}\,\big\{\mathrm{e}^t\,\big(\cos t + \sin t\big) - 1\big\}.$$

Example 4.5. Solve the following equation.

$$x(t) + \int_0^t \sinh(t - \tau)\,x'(\tau)\,\mathrm{d}\tau = \cosh t. \tag{4.11}$$

Solution: Differentiating (4.11), we have

$$x'(t) + \int_0^t \cosh(t - \tau)\,x'(\tau)\,\mathrm{d}\tau = \sinh t. \tag{4.12}$$

Further differentiating (4.12), we have

$$x''(t) + x'(t) + \int_0^t \sinh(t-\tau)\, x'(\tau)\, \mathrm{d}\tau = \cosh t. \tag{4.13}$$

Substituting (4.11) into (4.13), we have

$$x''(t) + x'(t) - x(t) = 0, \tag{4.14}$$

and rewrite (4.14) as follows.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \frac{-1+\sqrt{5}}{2} \right) \left(\frac{\mathrm{d}}{\mathrm{d}t} - \frac{-1-\sqrt{5}}{2} \right) x = 0. \tag{4.15}$$

Using Theorems 4.1 and 4.2, we can obtain the general solution to (4.14) or (4.15) as follows.

$$x(t) = C_1\, \mathrm{e}^{\frac{-1+\sqrt{5}}{2}t} + C_2\, \mathrm{e}^{\frac{-1-\sqrt{5}}{2}t}. \tag{4.16}$$

We also get $x(0) = 1$ and $x'(0) = 0$ by putting $t = 0$ to (4.11) and (4.12), and

$$x'(t) = \frac{-1+\sqrt{5}}{2}\, C_1\, \mathrm{e}^{\frac{-1+\sqrt{5}}{2}t} - \frac{1+\sqrt{5}}{2}\, C_2\, \mathrm{e}^{\frac{-1-\sqrt{5}}{2}t} \tag{4.17}$$

by differentiating (4.16). Substituting $t = 0$ into (4.16) and (4.17), we have

$$C_1 + C_2 = 1, \quad \frac{-1+\sqrt{5}}{2}\, C_1 - \frac{1+\sqrt{5}}{2}\, C_2 = 0 \quad \therefore C_1 = \frac{5+\sqrt{5}}{10}, \quad C_2 = \frac{-5+\sqrt{5}}{10}. \tag{4.18}$$

Thus, by substituting (4.18) into (4.16), we obtain the desired solution to (4.11) as follows.

$$x(t) = \frac{5+\sqrt{5}}{10}\, \mathrm{e}^{\frac{\sqrt{5}-1}{2}t} - \frac{5-\sqrt{5}}{10}\, \mathrm{e}^{-\frac{\sqrt{5}+1}{2}t}.$$

Example 4.6. Solve the following equation.

$$x(t) + \int_0^t \sin(t-\tau)\, x''(\tau)\, \mathrm{d}\tau = \cos t. \tag{4.19}$$

Solution: Differentiating (4.19), we have

$$x'(t) + \int_0^t \cos(t-\tau)\, x''(\tau)\, \mathrm{d}\tau = -\sin t. \tag{4.20}$$

Further differentiating (4.20), we have

$$2\, x''(t) - \int_0^t \sin(t-\tau)\, x''(\tau)\, \mathrm{d}\tau = -\cos t. \tag{4.21}$$

Substituting (4.19) into (4.21), we have

$$2\, x''(t) + x(t) = 0, \tag{4.22}$$

and rewrite (4.22) as follows.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}-\frac{\sqrt{-1}}{\sqrt{2}}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\sqrt{-1}}{\sqrt{2}}\right)x=0. \tag{4.23}$$

Using Theorems 4.1 and 4.2, we can obtain the general solution to (4.22) or (4.23) as follows.

$$x(t)=C_1\,\mathrm{e}^{\frac{\sqrt{-1}}{\sqrt{2}}t}+C_2\,\mathrm{e}^{-\frac{\sqrt{-1}}{\sqrt{2}}t}. \tag{4.24}$$

We also get $x(0)=1$ and $x'(0)=0$ by putting $t=0$ to (4.19) and (4.20), and

$$x'(t)=\frac{\sqrt{-1}}{\sqrt{2}}\,C_1\,\mathrm{e}^{\frac{\sqrt{-1}}{\sqrt{2}}t}-\frac{\sqrt{-1}}{\sqrt{2}}\,C_2\,\mathrm{e}^{-\frac{\sqrt{-1}}{\sqrt{2}}t} \tag{4.25}$$

by differentiating (4.24). Substituting $t=0$ into (4.24) and (4.25), we have

$$C_1+C_2=1,\quad \frac{\sqrt{-1}}{\sqrt{2}}\,C_1-\frac{\sqrt{-1}}{\sqrt{2}}\,C_2=0\quad \therefore C_1=C_2=\frac{1}{2}. \tag{4.26}$$

Thus, by substituting (4.26) into (4.24), we obtain the desired solution to (4.19) as follows.

$$x(t)=\frac{1}{2}\left(\mathrm{e}^{\frac{\sqrt{-1}}{\sqrt{2}}t}+\mathrm{e}^{-\frac{\sqrt{-1}}{\sqrt{2}}t}\right)=\cos\frac{\sqrt{2}\,t}{2}.$$

Example 4.7. Solve the following problem.

$$\begin{cases} x''(t)+x'(t)+x(t)+\displaystyle\int_0^t \sinh(t-\tau)\,\{\,x'(\tau)+x(\tau)\,\}\,\mathrm{d}\tau=\cosh t,\\ x(0)=x'(0)=0. \end{cases} \tag{4.27}$$

Solution: Differentiating (4.27), we have

$$x'''(t)+x''(t)+x'(t)+\int_0^t \cosh(t-\tau)\,\{\,x'(\tau)+x(\tau)\,\}\,\mathrm{d}\tau=\sinh t. \tag{4.28}$$

Further differentiating (4.28), we have

$$x^{(4)}(t)+x'''(t)+x''(t)+x'(t)+x(t)+\int_0^t \sinh(t-\tau)\,\{\,x'(\tau)+x(\tau)\,\}\,\mathrm{d}\tau=\cosh t. \tag{4.29}$$

Substituting the equation in (4.27) into (4.29), we have

$$x^{(4)}(t)+x'''(t)=0, \tag{4.30}$$

and rewrite (4.30) as follows.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^3\left(\frac{\mathrm{d}}{\mathrm{d}t}+1\right)x=0. \tag{4.31}$$

Using Theorems 4.1 and 4.2, we can obtain the general solution to (4.30) or (4.31) as follows.

$$x(t)=C_1+C_2\,t+C_3\,t^2+C_4\,\mathrm{e}^{-t}. \tag{4.32}$$

We also get

$$x''(0)=-x'(0)-x(0)+1=1,\quad x'''(0)=-x''(0)-x'(0)=0, \tag{4.33}$$

by putting $t = 0$ to the equation in (4.27) and (4.28), and

$$\begin{aligned} x'(t) &= C_2 + 2\,C_3\,t - C_4\,\mathrm{e}^{-t}, \\ x''(t) &= 2\,C_3 + C_4\,\mathrm{e}^{-t}, \quad x'''(t) = -C_4\,\mathrm{e}^{-t}, \end{aligned} \tag{4.34}$$

by differentiating (4.32). Noting (3.33) and substituting $t = 0$ into (4.32) and (4.34), we have

$$C_1 + C_4 = 0, \quad C_2 - C_4 = 0, \quad 2\,C_3 + C_4 = 1, \quad C_4 = 0 \quad \therefore C_1 = C_2 = C_4 = 0, \quad C_3 = \frac{1}{2}. \tag{4.35}$$

Thus, by substituting (4.35) into (4.32), we obtain the desired solution to (4.27) as follows.

$$x(t) = \frac{1}{2}\,t^2.$$

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