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Global Structure of the Diffusive Dispersive Contact Waves

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Abstract. In this paper, we consider the global structure of diffusive dispersive contact wave. The diffusive dispersive contact wave is the unique global in time solution to a Cauchy problem for the linear diffusive dispersive conservation law, where the far field states are prescribed. We introduce how to obtain the diffusive dispersive contact wave by applying the elementary Fourier analysis.

要旨：本論文では、消散的分散的接触波の大域構造について考える。消散的分散的接触波とは、消散的分散的保存則の遠方条件付きCauchy問題の一意的時間大域解のことである。この消散的分散的接触波が、初等的なFourier解析を援用することで如何にして得られるかについて紹介する。

1. Introduction and main theorems.

We consider the Cauchy problem for the linear diffusive dispersive conservation law

$$\begin{cases} \partial_t u + \partial_x \big(\lambda\, u - \mu\, \partial_x u + \delta\, \partial_x^2 u \big) = 0 \quad \big(t > 0,\ x \in \mathbb{R} \big), \\ u(0,x) = u_0^{\mathrm{R}}(\, x\, ;\, u_-,\ u_+) := \begin{cases} u_- & (x<0), \\ u_+ & (x>0), \end{cases} \\ \displaystyle\lim_{x \to \pm\infty} u(t,x) = u_\pm \quad \big(t \geq 0 \big), \end{cases} \tag{1.1}$$

where, $u(t,x)$ is the unknown function of $t>0$ and $x \in \mathbb{R}$, the so-called conserved quantity,

$$\lambda\, u - \mu\, \partial_x u + \delta\, \partial_x^2 u \quad \big(\mu > 0,\ \delta,\ \lambda \in \mathbb{R} \big)$$

is the total flux (that is, the functions $\lambda\, u$, $\mu\, \partial_x u$ and $\delta\, \partial_x^2 u$ stand for the convective flux, viscous/diffusive one and dispersive one, respectively), $u_0^{\mathrm{R}}(x) = u_0^{\mathrm{R}}(\, x\, ;\, u_-,\ u_+)$ is the initial data which is the so-called Riemann data, and $u_\pm \in \mathbb{R}$ are the prescribed far field states.

It is known that if $\delta = 0$, then (1.1) becomes the Cauchy problem for the linear heat convective equation

$$\begin{cases} \partial_t u + \partial_x \big(\lambda\, u - \mu\, \partial_x u \big) = 0 \quad \big(t > 0,\ x \in \mathbb{R} \big), \\ u(0,x) = u_0^{\mathrm{R}}(\, x\, ;\, u_-,\ u_+) = \begin{cases} u_- & (x<0), \\ u_+ & (x>0), \end{cases} \\ \displaystyle\lim_{x \to \pm\infty} u(t,x) = u_\pm \quad \big(t \geq 0 \big), \end{cases} \tag{1.2}$$

and the solution to (1. 2) is the well-known viscous contact wave connecting u_- to u_+, which has the form

$$u(t,x) = U^{\mathrm{V,C}} \left(\frac{x - \lambda\, t}{\sqrt{t}}\, ;\, u_-,\ u_+ \right) = u_- + \frac{u_+ - u_-}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-\lambda t}{\sqrt{4\mu t}}} \mathrm{e}^{-\eta^2}\, \mathrm{d}\eta. \tag{1.3}$$

We also recall that the viscous contact wave (1.3) is closely related and corresponded to the contact discontinuity. The contact discontinuity is a travelling wave and unique weak solution to the Riemann problem for the linear convctive equation

$$\begin{cases} \partial_t u + \lambda\,\partial_x u = 0 & \big(t > 0,\ x \in \mathbb{R}\big), \\ u(0,x) = u_0^{\mathrm{R}}(x) = \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0), \end{cases} \end{cases} \tag{1.4}$$

which has the form

$$u(t,x) = u^{\mathrm{C}}(\,x - \lambda\,t\,;\,u_-,\,u_+) := \begin{cases} u_- & (x < \lambda\,t), \\ u_+ & (x > \lambda\,t). \end{cases}$$

We are interested in the exact form of the solution to (1.1). The solution to (1.1) is the so-called diffusive dispersive contact wave and also corresponded to the contact discontinuity of (1.4). The form of the diffusive dispersive contact wave is given in the following main theorem.

Theorem 1.1. *Let $\mu > 0$, δ, λ, $u_\pm \in \mathbb{R}$. The Cauchy problem* (1.1) *has a unique global in time solution, that is, the diffusive dispersive contact wave, connecting u_- to u_+, which has the form*

$$u(t,x) = U^{\mathrm{D,D,C}}\left(\frac{x - \lambda\,t}{\sqrt{t}}\,;\,u_-,\,u_+\right),$$

where

$$\begin{aligned} &U^{\mathrm{D,D,C}}\left(\frac{x}{\sqrt{t}}\,;\,u_-,\,u_+\right) \\ &= \frac{3\,\delta\,t}{\pi}\left(u_- \int_{\frac{x}{\sqrt{4\mu t}}}^{\infty} + u_+ \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}}\right) \mathrm{e}^{-\eta^2} \int_0^{\eta} \mathrm{e}^{\zeta^2} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t} \cos\left(\delta\,\xi^3\,t - \zeta\,\xi\,\sqrt{4\,\mu\,t}\right) \mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &\quad + \sqrt{\frac{\mu\,t}{\pi}}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t} \cos\big(\,\delta\,\xi^3\,t\,\big)\,\mathrm{d}\xi\right)\left(u_- + \frac{u_+ - u_-}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}} \mathrm{e}^{-\eta^2}\,\mathrm{d}\eta\right). \end{aligned}$$

The proof of Therem 1.1 is given in Section 2.

Remark 1.2. We note that the diffussive dispersive contact wave in Theorem 1.1 becomes the viscous contact wave (1.4) when $\delta = 0$, that is,

$$U^{\mathrm{D,D,C}}\left(\frac{x - \lambda\,t}{\sqrt{t}}\,;\,u_-,\,u_+\right)\bigg|_{\delta=0} = U^{\mathrm{V,C}}\left(\frac{x - \lambda\,t}{\sqrt{t}}\,;\,u_-,\,u_+\right).$$

Remark 1.3. It is noted that if $\lambda\,u \mapsto f(u)$ (more general case), then the equation in (1.1) becomes the generalized Korteweg-de Vries Burgers equation

$$\partial_t u + \partial_x\big(\,f(u) - \mu\,\partial_x u + \delta\,\partial_x^2 u\,\big) = 0. \tag{1.5}$$

We also note that there are many results conserning with the asymptotic stabilities of (1.5) (see [2, 6, 24, 27, 37] and so on, cf. [1, 3, 4, 5, 7-23, 25, 26, 28-36, 38, 39]).

Some Notation. $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the usual Fourier transform and inverse Fourier transform, and defined by

$$\hat{v}(\xi) = \mathcal{F}[\,v\,](\xi) := \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\sqrt{-1}x\xi}\,v(x)\,\mathrm{d}x, \quad \check{v}(x) = \mathcal{F}^{-1}[\,v\,](x) := \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\sqrt{-1}x\xi}\,v(\xi)\,\mathrm{d}\xi,$$

respectively.

2. Diffusive dispersive contact wave.

In this section, we give a sketch of the proof of main Theorem 1.1 with the case $\lambda = 0$ for simplicity. Because the proof of the uniqueness of the diffusive dispersive contact wave is standard, we omit it.

We rewrite our problem with the case $\lambda = 0$ as follows.

$$\begin{cases} \partial_t u + \partial_x\big(-\mu\,\partial_x u + \delta\,\partial_x^2 u\big) = 0 \quad \big(t > 0,\ x \in \mathbb{R}\big), \\ u(0,x) = u_0^{\mathrm{R}}(\,x\,;\,u_-,\ u_+) = \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0), \end{cases} \\ \lim\limits_{x\to\pm\infty} u(t,x) = u_\pm \quad \big(t \geq 0\big), \end{cases} \tag{2.1}$$

where $\mu > 0,\ \delta,\ u_\pm \in \mathbb{R}$. By using the Fourier transform formally, (2.1) becomes

$$\begin{cases} \partial_t \hat{u} = -\big(\mu\,\xi^2 + \sqrt{-1}\,\delta\,\xi^3\big)\,\hat{u} \quad \big(t \geq 0,\ \xi \in \mathbb{R}\big), \\ \hat{u}(0,\xi) = \widehat{u_0^{\mathrm{R}}}(\xi) \quad \big(\xi \in \mathbb{R}\big). \end{cases} \tag{2.2}$$

From (2.2), we easily get as

$$\hat{u}(t,\xi) = \widehat{u_0^{\mathrm{R}}}(\xi)\,\mathrm{e}^{-(\mu\xi^2+\sqrt{-1}\delta\xi^3)t}. \tag{2.3}$$

By using the inverse Fourier transform, (2.3) becomes

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}\mathrm{e}^{-(\mu\xi^2+\sqrt{-1}\delta\xi^3)t}\mathcal{F}u_0^{\mathrm{R}}(x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\sqrt{-1}x\xi} \int_{-\infty}^{\infty} \mathrm{e}^{-(\mu\xi^2+\sqrt{-1}\delta\xi^3)t-\sqrt{-1}y\xi}\,u_0^{\mathrm{R}}(y)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t-\sqrt{-1}\left(\delta\xi^3 t+(y-x)\xi\right)}\,u_0^{\mathrm{R}}(y)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0^{\mathrm{R}}(y)\int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t-\sqrt{-1}\left(\delta\xi^3 t+(y-x)\xi\right)}\,\mathrm{d}\xi\,\mathrm{d}y. \end{aligned} \tag{2.4}$$

Putting

$$I(t,y-x) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t-\sqrt{-1}\left(\delta\xi^3 t+(y-x)\xi\right)}\,\mathrm{d}\xi,$$

we immediately have

$$I(t,y-x) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\delta\,\xi^3\,t + (y-x)\,\xi\big)\,\mathrm{d}\xi. \tag{2.5}$$

Differentiating (2.5) with respect to y, we obtain the following partial differential equation.

$$\partial_y I + \frac{y-x}{2\,\mu\,t}\,I + \frac{3\,\delta}{2\,\mu}\int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\delta\,\xi^3\,t + (y-x)\,\xi\big)\,\mathrm{d}\xi = 0. \tag{2.6}$$

Multiplying (2.6) by

$$\mathrm{e}^{\int \frac{y-x}{2\mu t}\,\mathrm{d}y} = \mathrm{e}^{\frac{(y-x)^2}{4\mu t}+\frac{C(t,x)}{2\mu t}},$$

where $C = C(t,x)$ is arbitrary taken, we immediately have

$$\partial_y\left(\mathrm{e}^{\int \frac{y-x}{2\mu t}\,\mathrm{d}y}\,I\right) = \frac{-3\,\delta}{2\,\mu}\,\mathrm{e}^{\int \frac{y-x}{2\mu t}\,\mathrm{d}y}\int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\delta\,\xi^3\,t + (y-x)\,\xi\big)\,\mathrm{d}\xi \tag{2.7}$$

and easily get by integrating (2.7) with respect to y that the solution to (2.6) as follows.

$$
\begin{aligned}
I(t,y-x) = {} & \frac{-3\,\delta}{2\,\mu}\,\mathrm{e}^{-\frac{(y-x)^2}{4\mu t}} \int_x^y \mathrm{e}^{\frac{(s-x)^2}{4\mu t}} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t + (s-x)\,\xi\,\big)\,\mathrm{d}\xi\,\mathrm{d}s \\
& + \mathrm{e}^{-\frac{(y-x)^2}{4\mu t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t\,\big)\,\mathrm{d}\xi.
\end{aligned} \tag{2.8}
$$

Substituting (2.8) into (2.4), we have

$$
\begin{aligned}
u(t,x) = {} & \frac{-3\,\delta}{2\,\mu} \int_{-\infty}^{\infty} u_0^{\mathrm{R}}(y)\,\mathrm{e}^{-\frac{(y-x)^2}{4\mu t}} \int_x^y \mathrm{e}^{\frac{(s-x)^2}{4\mu t}} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t + (s-x)\,\xi\,\big)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}y \\
& + \frac{1}{2\,\pi}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t\,\big)\,\mathrm{d}\xi\right)\left(\int_{-\infty}^{\infty} u_0^{\mathrm{R}}(y)\,\mathrm{e}^{-\frac{(y-x)^2}{4\mu t}}\,\mathrm{d}y\right).
\end{aligned} \tag{2.9}
$$

Noting the definition of u_0^{R}, separating the integral region, defining a new variable $\eta = -(y-x)/\sqrt{4\,\mu\,t}$ and using η, the first term on the right-hand side of (2.9) becomes

$$
\begin{aligned}
& \int_{-\infty}^{\infty} u_0^{\mathrm{R}}(y)\,\mathrm{e}^{-\frac{(y-x)^2}{4\mu t}} \int_x^y \mathrm{e}^{\frac{(s-x)^2}{4\mu t}} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t + (s-x)\,\xi\,\big)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}y \\
&= \left(u_- \int_{-\infty}^{0} + u_+ \int_0^{\infty}\right)\mathrm{e}^{-\frac{(y-x)^2}{4\mu t}} \int_x^y \mathrm{e}^{\frac{(s-x)^2}{4\mu t}} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t + (s-x)\,\xi\,\big)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}y \\
&= \sqrt{4\,\mu\,t}\left(u_- \int_{\frac{x}{\sqrt{4\mu t}}}^{\infty} + u_+ \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}}\right) \\
&\qquad \mathrm{e}^{-\eta^2} \int_x^{x-\eta\sqrt{4\mu t}} \mathrm{e}^{\frac{(s-x)^2}{4\mu t}} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t + (s-x)\,\xi\,\big)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}\eta.
\end{aligned} \tag{2.10}
$$

Similarly, the second term on the right-hand side of (2.9) becomes

$$
\begin{aligned}
\int_{-\infty}^{\infty} u_0^{\mathrm{R}}(y)\,\mathrm{e}^{-\frac{(y-x)^2}{4\mu t}}\,\mathrm{d}y &= \left(u_- \int_{-\infty}^{0} + u_+ \int_0^{\infty}\right)\mathrm{e}^{-\frac{(y-x)^2}{4\mu t}}\,\mathrm{d}y \\
&= \sqrt{4\,\mu\,t}\left(u_- \int_{\frac{x}{\sqrt{4\mu t}}}^{\infty} + u_+ \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}}\right)\mathrm{e}^{-\eta^2}\,\mathrm{d}\eta \\
&= \sqrt{4\,\mu\,t}\left(u_-\left(\int_{-\infty}^{\infty} - \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}}\right) + u_+ \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}}\right)\mathrm{e}^{-\eta^2}\,\mathrm{d}\eta \\
&= \sqrt{4\,\pi\,\mu\,t}\left(u_- + \frac{u_+ - u_-}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}} \mathrm{e}^{-\eta^2}\,\mathrm{d}\eta\right).
\end{aligned} \tag{2.11}
$$

Defining a new variable $\zeta = -(s-x)/\sqrt{4\,\mu\,t}$ and using ζ, (2.10) further becomes

$$
\begin{aligned}
& \int_x^{x-\eta\sqrt{4\mu t}} \mathrm{e}^{\frac{(s-x)^2}{4\mu t}} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t + (s-x)\,\xi\,\big)\,\mathrm{d}\xi\,\mathrm{d}s \\
&= -\sqrt{4\,\mu\,t} \int_0^{\eta} \mathrm{e}^{\zeta^2} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t - \zeta\,\xi\,\sqrt{4\,\mu\,t}\,\big)\,\mathrm{d}\xi\,\mathrm{d}\zeta.
\end{aligned} \tag{2.12}
$$

Substituting (2.10)-(2.12) into (2.9), we obtain the desired diffusive dispersive contact wave, that is,

$$
\begin{aligned}
& u(t,x) \\
&= \frac{3\,\delta\,t}{\pi}\left(u_- \int_{\frac{x}{\sqrt{4\mu t}}}^{\infty} + u_+ \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}}\right)\mathrm{e}^{-\eta^2} \int_0^{\eta} \mathrm{e}^{\zeta^2} \int_{-\infty}^{\infty} \xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\Big(\,\delta\,\xi^3\,t - \zeta\,\xi\,\sqrt{4\,\mu\,t}\,\Big)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\eta \\
&\quad + \sqrt{\frac{\mu\,t}{\pi}}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\,\delta\,\xi^3\,t\,\big)\,\mathrm{d}\xi\right)\left(u_- + \frac{u_+ - u_-}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}} \mathrm{e}^{-\eta^2}\,\mathrm{d}\eta\right).
\end{aligned} \tag{2.13}
$$

Thus, Theorem 1.1 is proved.

Remark 2.1. We note that the proof in Section 2 is based on very formal calculation. Therefore, the process of the proof should be mathematically justified. To do that, we differentiate (2.1) with respect to x, put $v=\partial_x u$ and consider the following problem.

$$\begin{cases} \partial_t v+\partial_x\big(-\mu\,\partial_x v+\delta\,\partial_x^2 v\big)=0 \quad \big(t>0,\ x\in\mathbb{R}\big), \\ v(0,x)=\delta(x) \quad (x\in\mathbb{R}), \end{cases} \tag{2.14}$$

where $\delta(x)$ is the Dirac δ-distribution. By the similar arguments as (2.2)-(2.9), we can get the solution to (2.14) as follows.

$$\begin{aligned} v(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(y)\,I(t,y-x)\,\mathrm{d}y \\ &= \frac{1}{2\pi}\;I(t,y-x)\big|_{y=0} \\ &= \frac{-3\,\delta}{4\,\pi\,\mu}\,\mathrm{e}^{-\frac{x^2}{4\mu t}}\int_x^0 \mathrm{e}^{\frac{(s-x)^2}{4\mu t}}\int_{-\infty}^{\infty}\xi^2\,\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\delta\,\xi^3\,t+(s-x)\,\xi\big)\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad+\frac{1}{2\,\pi}\,\mathrm{e}^{-\frac{x^2}{4\mu t}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mu\xi^2 t}\,\cos\big(\delta\,\xi^3\,t\big)\,\mathrm{d}\xi. \end{aligned} \tag{2.15}$$

Therefore, by using (2.15), we can obtain (2.13) from

$$u(t,x)=u_-+(u_+-u_-)\int_{-\infty}^{x}v(t,y)\,\mathrm{d}y.$$

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References

[1] K. Andreiev, I. Egorova, T.L. Lange and G. Teschl, Rarefaction waves of the Korteweg-de Vries equation via nonlinear steepest deecent, J. Differential Equations **10** (2016), pp. 5371-5410.

[2] J.L. Bona and M.E. Schonbek, Travelling-wave solutions to the Korteweg-deVries-Burgers equation, Proc. Roy. Soc. Edinburgh **101A** (1985), pp. 207-226.

[3] J.L. Bona, S.V. Rajopadhye and M.E. Schonbek, Models for the propagation of bores I. Two dimensional theory, Differential Integral Equations **7** (1994), pp. 699-734.

[4] S. Deng and W. Wang, Pointwise decaying rate of large perturbation around viscous shock for scalar viscous conservation law, Sci. Chin. Math. **56** (2013), pp. 729-736.

[5] R. Duan, L.-L. Fan, J.-S. Kim and L.-Q. Xie, Nonlinear stability of strong rarefaction waves for the generalized KdV-Burgers-Kuramoto equation with large initial perturbation, Nonlinear Anal. **73** (2010), pp. 3254-3267.

[6] R. Duan and H.-J. Zhao, Global stability of strong rarefaction waves for the generalized KdV-Burgers equation, Nonlinear Anal. **66** (2007), pp. 1100-1117.

[7] I. Egorova, K. Grunert and G. Teschl, On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data I. Schwartz-type perturbations, Nonlinearity **22** (2009), pp. 1431-1457.

[8] I. Egorova and G. Teschl, On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data 2. Perturbations with finite moments, J. Anal. Math. **115** (2011), pp. 71-101.

[9] H. Freistühler and D. Serre, L^1-stability of shock waves in scalar viscous conservation laws, Comm. Pure

Appl. Math. **51** (1998), pp. 291-301.

[10] E. Harabetian, Rarefactions and large time behavior for parabolic equation and monotone schemes, Comm. Math. Phys. **114** (1988), pp. 527-536.

[11] I. Hashimoto, Y. Ueda and S. Kawashima, Convergence rate to the nonlinear waves for viscous conservation laws on the half line, Methods Appl. Anal. **16** (2009), pp. 389-402.

[12] I. Hashimoto and A. Matsumura, Large time behavior of solutions to an initial boundary value problem on the half space for scalar viscous conservation law, Methods Appl. Anal. **14** (2007), pp. 45-59.

[13] Y. Hattori and K. Nishihara, A note on the stability of rarefaction wave of the Burgers equation, Japan J. Indust. Appl. Math. **8** (1991), pp. 85-96.

[14] A. Kurganov, D. Levy and P. Rosenau, On Burgers-type equations with nonmonotonic dissipative fluxes, Comm. Pure Appl. Math. **51** (1998), pp. 443-473.

[15] A.M. Il'in, A.S. Kalašnikov and O.A. Oleĭnik, Second-order linear equations of parabolic type, Uspekhi Math. Nauk SSSR **17** (1962), pp. 3-146 (in Russian); Russ. Math. Surv. **17** (1962), pp. 1-143 (in English).

[16] A.M. Il'in and O.A. Oleĭnik, Asymptotic behavior of the solutions of the Cauchy problem for some quasi-linear equations for large values of the time, Mat. Sb. **51** (1960), pp. 191-216 (in Russian).

[17] P.D. Lax, Hyperbolic systems of conservation laws Ⅱ, Comm. Pure Appl. Math. **10** (1957), pp. 537-566.

[18] T.-P. Liu, A. Matsumura and K. Nishihara, Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves, SIAM J. Math. Anal. **29** (1998), pp. 293-308.

[19] A. Matsumura, Waves in compressible fuids: viscous shock, rarefaction and contact waves, in Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer-Verlag, New York, 2018.

[20] A. Matsumura and K. Nishihara, Asymptotics toward the rarefaction wave of the solutions of Burgers' equation with nonlinear degenerate viscosity, Nonlinear Anal. **23** (1994), pp. 605-614.

[21] A. Matsumura and K. Nishihara, Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity, Comm. Math. Phys. **165** (1994), pp. 83-96.

[22] A. Matsumura and N. Yoshida, Asymptotic behavior of solutions to the Cauchy problem for the scalar viscous conservation law with partially linearly degenerate flux, SIAM J. Math. Anal. **44** (2012), pp. 2526-2544.

[23] A. Matsumura and N. Yoshida, Global asymptotics toward the rarefaction waves for solutions to the Cauchy problem of the scalar conservation law with nonlinear viscosity. Osaka J. Math. **57** (2020), pp. 187–205.

[24] K. Nishihara and S.V. Rajopadhye, Asymptotic behavior of solutions to the Korteweg-deVries-Burgers equation. Differential Integral Equations **11** (1998), pp. 85-93.

[25] S. Osher and J. Ralston, L^1 stability of traveling waves with applications to convective porous media flow, Comm. Pure Appl. Math. **35** (1982), pp. 737-751.

[26] L.-Z. Ruan, W.-L. Gao and J. Chen, Asymptotic stability of the rarefaction wave for the generalized KdV-Burgers-Kuramoto equation. Nonlinear Anal. **68** (2008), pp. 402-411.

[27] Z.-A. Wang and C.-J. Zhu, Stability of the rarefaction wave for the generalized KdV-Burgers equation, Acta Math. Sci. **22B** (3) (2002), pp. 319-328.

[28] N. Yoshida, Decay properties of solutions toward a multiwave pattern for the scalar viscous conservation law with partially linearly degenerate flux, Nonlinear Anal. **96** (2014), pp. 189-210.

[29] N. Yoshida, Decay properties of solutions to the Cauchy problem for the scalar conservation law with nonlinearly degenerate viscosity, Nonlinear Anal. **128** (2015), pp. 48-76.

[30] N. Yoshida, Large time behavior of solutions toward a multiwave pattern for the Cauchy problem of the

scalar conservation law with degenerate flux and viscosity, in: Mathematical Analysis in Fluid and Gas Dynamics, Sūrikaisekikenkyūsho Kōkyūroku **1947** (2015), pp. 205-222.

[31] N. Yoshida, Asymptotic behavior of solutions toward a multiwave pattern for the scalar conservation law with the Ostwald-de Waele-type viscosity, SIAM J. Math. Anal. **49** (2017), pp. 2009-2036.

[32] N. Yoshida, Decay properties of solutions toward a multiwave pattern to the Cauchy problem for the scalar conservation law with degenerate flux and viscosity, J. Differential Equations **263** (2017), pp. 7513-7558.

[33] N. Yoshida, Asymptotic behavior of solutions toward the viscous shock waves to the Cauchy problem for the scalar conservation law with nonlinear flux and viscosity, SIAM J. Math. Anal. **50** (2018), pp. 891-932.

[34] N. Yoshida, Asymptotic behavior of solutions toward the rarefaction waves to the Cauchy problem for the scalar diffusive dispersive conservation laws, Nonlinear Anal. **189** (2019), pp. 1-19.

[35] N. Yoshida, Global structure of solutions toward the rarefaction waves for the Cauchy problem of the scalar conservation law with nonlinear viscosity, J. Differential Equations **269** (2020), no. 11, pp. 10350–10394.

[36] N. Yoshida, Asymptotic behavior of solutions toward a multiwave pattern to the Cauchy problem for the dissipative wave equation with partially linearly degenerate flux, Funkcial. Ekvac. **64** (2021), pp. 49–73.

[37] N. Yoshida, Asymptotic behavior of solutions to the rarefaction problem for the generalized Korteweg-de Vries Burgers equation, Bull. Fac. Edu., Univ. of Yamanashi **32** (2022), pp. 135 – 150.

[38] N. Yoshida, Asymptotic behavior of solutions toward the constant state to the Cauchy problem for the non-viscous diffusive dispersive conservation law, Preprint, arXiv:2107.07874.

[39] N. Yoshida, Global asymptotic stability of the rarefaction waves to the Cauchy problem for the scalar non-viscous diffusive dispersive conservation laws, Preprint.

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