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Asymptotic behavior of the Diffusive Dispersive Contact Waves

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Abstract. In this paper, we consider the asymptotic behavior of diffusive dispersive contact wave. We expect that the diffusive dispersive contact wave is structurally is similar to the viscous contact wave. In fact, we can show by the energy methods that the diffusive dispersive contact wave corresponds asymptotically to the viscous contact wave.

要旨：本論文では、消散的分散的接触波の大域構造について考える。消散的分散的接触波は構造的に粘性的接触波と同様であると期待される。実際に、エネルギー法により、この消散的分散的接触波は粘性的接触波に漸近的に一致することを示すことが出来る。

1. Introduction and main theorems.

We consider the Cauchy problem for the linear diffusive dispersive conservation law

$$\begin{cases} \partial_t u + \partial_x\big(\lambda u - \mu\,\partial_x u + \delta\,\partial_x^2 u\big) = 0 \quad \big(t>0,\ x\in\mathbb{R}\big), \\ u(0,x) = u_0^{\mathrm{R}}(\,x\,;\,u_-,\ u_+) := \begin{cases} u_- & (x<0), \\ u_+ & (x>0), \end{cases} \\ \displaystyle\lim_{x\to\pm\infty} u(t,x) = u_\pm \quad \big(t\ge 0\big), \end{cases} \tag{1.1}$$

where, $u(t,x)$ is the unknown function of $t>0$ and $x\in\mathbb{R}$, the so-called conserved quantity,

$$\lambda u - \mu\,\partial_x u + \delta\,\partial_x^2 u \quad \big(\mu>0,\ \delta,\ \lambda\in\mathbb{R}\big)$$

is the total flux (that is, the functions λu, $\mu\,\partial_x u$, $\delta\,\partial_x^2 u$ and $\delta\,\partial_x^2 u$ stand for the convective flux, viscous/diffusive one and dispersive one, respectively), $u_0^{\mathrm{R}}(x) = u_0^{\mathrm{R}}(\,x\,;\,u_-,\ u_+)$ is the initial data which is the so-called Riemann data, and $u_\pm\in\mathbb{R}$ are the prescribed far field states.

According to Yoshida [27], the exact solution to (1.1), that is, the diffusive dispersive contact wave connecting u_- to u_+, is given by

$$\begin{aligned} u(t,x) &= U^{\mathrm{D,D,C}}\left(\frac{x-\lambda t}{\sqrt{t}}\,;\,u_-,\ u_+\right) \\ &= \frac{3\,\delta\,t}{\pi}\left(u_-\int_{\frac{x-\lambda t}{\sqrt{4\mu t}}}^{\infty} + u_+\int_{-\infty}^{\frac{x-\lambda t}{\sqrt{4\mu t}}}\right)\mathrm{e}^{-\eta^2}\int_0^{\eta}\mathrm{e}^{\zeta^2}\int_{-\infty}^{\infty}\xi^2\,\mathrm{e}^{-\mu\xi^2 t}\cos\Big(\delta\,\xi^3\,t - \zeta\,\xi\,\sqrt{4\,\mu\,t}\Big)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &\quad + \sqrt{\frac{\mu\,t}{\pi}}\left(\int_{-\infty}^{\infty}\mathrm{e}^{-\mu\xi^2 t}\cos\big(\delta\,\xi^3\,t\big)\,\mathrm{d}\xi\right)\left(u_- + \frac{u_+-u_-}{\sqrt{\pi}}\int_{-\infty}^{\frac{x-\lambda t}{\sqrt{4\mu t}}}\mathrm{e}^{-\eta^2}\,\mathrm{d}\eta\right). \end{aligned} \tag{1.2}$$

Also it is known that the viscous contact wave connecting u_- to u_+ is the exact solution to the Cauchy problem for the linear heat convective equation

$$\begin{cases} \partial_t u + \partial_x\big(\lambda u - \mu\,\partial_x u\big) = 0 \quad \big(t > 0,\ x \in \mathbb{R}\big), \\ u(0,x) = u_0^{\mathrm{R}}(\,x\,;\,u_-,\,u_+) = \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0), \end{cases} \\ \displaystyle\lim_{x\to\pm\infty} u(t,x) = u_\pm \quad \big(t \geq 0\big), \end{cases} \tag{1.3}$$

which has the form

$$u(t,x) = U^{\mathrm{V,C}}\left(\frac{x-\lambda t}{\sqrt{t}}\,;\,u_-,\,u_+\right) = u_- + \frac{u_+ - u_-}{\sqrt{\pi}}\int_{-\infty}^{\frac{x-\lambda t}{\sqrt{4\mu t}}} \mathrm{e}^{-\eta^2}\,\mathrm{d}\eta \tag{1.4}$$

(see [10, 15, 17, 18, 23]).

It is easily see that (1.2) immediately becomes (1.4) by putting $\delta = 0$, and (1.2) and (1.4) are corresponded to the contact discontinuity (travelling wave solution)

$$u(t,x) = u^{\mathrm{C}}(\,x-\lambda t\,;\,u_-,\,u_+) := \begin{cases} u_- & (x < \lambda t), \\ u_+ & (x > \lambda t), \end{cases} \tag{1.5}$$

to the Riemann problem for the following linear convctive equation (see [5] and so on).

$$\begin{cases} \partial_t u + \lambda\,\partial_x u = 0 \qquad \big(t > 0,\ x \in \mathbb{R}\big), \\ u(0,x) = u_0^{\mathrm{R}}(x) := \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0). \end{cases} \end{cases} \tag{1.6}$$

We are interested in the asymptotic bahavior of the diffusive dispersive contact wave (1.2). We expect that (1.2) tends toward (1.4) as time goes to infinity from the relation to (1.2), (1.4) and (1.5).

We are ready to state our main theorem.

Theorem 1.1. *Let* $\mu > 0,\ \delta,\ \lambda,\ u_\pm \in \mathbb{R}$. *The diffusive dispersive contact wave* (1.2) *tends uniformly in* x *toward the viscous contact wave* (1.4) *as time goes to infinity, that is,*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left| U^{\mathrm{D,D,C}}\left(\frac{x-\lambda t}{\sqrt{t}}\,;\,u_-,\,u_+\right) - U^{\mathrm{V,C}}\left(\frac{x-\lambda t}{\sqrt{t}}\,;\,u_-,\,u_+\right)\right| = 0.$$

We give the proof of Therem 1.1 in Section 3 for an essential case $\lambda = 0$.

This paper is organized as follows. In Section 2, we prepare the basic properties of the viscous contact wave (1.4) for an essential case $\lambda = 0$, for the proof of the main theorem. In Section 3, we reformulate the problem (1.1) in terms of the deviation from the viscous contact wave (1.4), and finally establish the desired asymptotic stability.

Some Notation. We denote by C generic positive constants unless they need to be distinguished. In particular, use $C_{\alpha,\beta,\cdots}$ when we emphasize the dependency on $\alpha,\ \beta,\ \cdots$.
For function spaces, $L^p = L^p(\mathbb{R})$ and $H^k = H^k(\mathbb{R})$ denote the usual Lebesgue space and *k*-th order Sobolev space on the whole space $\mathbb{R}$ with norms $||\cdot||_{L^p}$ and $||\cdot||_{H^k}$, respectively.

2. Viscous contact wave.

In this section, we shall arrange a lemma concerning with the basic properties of the viscous contact wave for accomplishing the proof of the main theorem.

The properties of the viscous contact wave for the case $\lambda = 0$, that is,

$$U^{\mathrm{V,C}}\left(\frac{x}{\sqrt{t}}\,;\,u_-,\,u_+\right) = u_- + \frac{u_+ - u_-}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}} \mathrm{e}^{-\eta^2}\,\mathrm{d}\eta, \tag{2.1}$$

are stated in the next lemma.

Lemma 2.1. *Let $\mu > 0,\ u_{\pm} \in \mathbb{R}$. we have the following properties:*
(1) $U^{\mathrm{V,C}}$ *defined by* (1.4) *is* B^{∞} *on* $(0,\ \infty) \times \mathbb{R}$, *and a self-similar solution of the Cauchy problem*

$$\begin{cases} \partial_t U^{\mathrm{V,C}} - \mu\, \partial_x^2 U^{\mathrm{V,C}} = 0 \quad \big(t > 0,\ x \in \mathbb{R} \big), \\ U^{\mathrm{V,C}}(0,x) = u_0^{\mathrm{R}}(\, x\,;\ u_-,\ u_+) = \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0), \end{cases} \\ \lim\limits_{x \to \pm\infty} U^{\mathrm{V,C}}(t,x) = u_{\pm} \quad \big(t \geq 0 \big). \end{cases}$$

(2) $\min\{u_-, u_+\} < U^{\mathrm{V,C}}(t,x) < \max\{u_-, u_+\}$ and $\partial_x U^{\mathrm{V,C}}(t,x) > 0$ on $(0,\ \infty) \times \mathbb{R}$.
(3) *It follows that for any* $1 \leq q \leq \infty$,

$$\big\|\, \partial_x U^{\mathrm{V,C}}(t) \,\big\|_{L^q} \leq \frac{|\, u_+ - u_- \,|}{(\, 4\,\pi\,\mu\,)^{\frac{1}{2}(1-\frac{1}{q})}}\, t^{-\frac{1}{2}(1-\frac{1}{q})} \qquad (t > 0).$$

(4) *It follows that for any* $1 \leq q \leq \infty$,

$$\big\|\, \partial_x^2 U^{\mathrm{V,C}}(t) \,\big\|_{L^q} \leq \frac{2^{1+\frac{1}{q}} |\, u_+ - u_- \,|}{(\, 4\,\mu\,)^{1-\frac{1}{2q}}\, \pi^{\frac{1}{2}}} \left(\int_0^{\infty} y^q\, \mathrm{e}^{-y^{2q}}\, \mathrm{d}y \right)^{\frac{1}{q}} t^{-\left(1-\frac{1}{2q}\right)} \qquad (t > 0).$$

(5) *It follows that for any* $1 \leq q \leq \infty$,

$$\big\|\, \partial_x^3 U^{\mathrm{V,C}}(t) \,\big\|_{L^q} \leq \frac{2^{1+\frac{1}{q}} |\, u_+ - u_- \,|}{(\, 4\,\mu\,)^{\frac{3}{2}-\frac{1}{2q}}\, \pi^{\frac{1}{2}}} \left(\int_0^{\infty} \big(\, 1 + 2^q\, y^{2q} \,\big)\, \mathrm{e}^{-y^{2q}}\, \mathrm{d}y \right)^{\frac{1}{q}} t^{-\left(\frac{3}{2}-\frac{1}{2q}\right)} \qquad (t > 0).$$

Because the proofs of (1)-(4) are well-known and (5) is immediately obtained by

$$\partial_x^3 U^{\mathrm{V,C}}(t) = \frac{-2\,|\, u_+ - u_- \,|}{(\, 4\,\mu\, t\,)^{\frac{3}{2}}\, \pi^{\frac{1}{2}}}\, \mathrm{e}^{-\frac{x^2}{4\mu t}} \left(1 - \frac{x^2}{4\,\mu\, t} \right)$$

from (2.1), we thus omit the proofs here (see [10, 15, 17, 18, 23]).

3. Reformulation of the problem.

In this section, we reformulate our Cauchy problem (1.1) in terms of the deviation from the asymptotic state, viscous contact wave (1.4).

Now we write $U^{\mathrm{V,C}}(1+t, x)$ and $U^{\mathrm{D,C,C}}(1+t, x)$ again $U^{\mathrm{V,C}}(t,x)$ and $U^{\mathrm{D,C,C}}(t,x)$, respectively, for simplicity, put

$$U^{\mathrm{D,C,C}}(t,x) = U^{\mathrm{V,C}}(t,x) + \phi(t,x), \tag{3.1}$$

and reformulate our problem (1.1) by using (3.1) and (1) in Lemma 2.1 as follows.

$$\begin{cases} \partial_t \phi + \partial_x \big(-\mu\, \partial_x \phi + \delta\, \partial_x^2 \phi \big) = -\delta\, \partial_x^3 U^{\mathrm{V,C}} \quad \big(t > 0,\ x \in \mathbb{R} \big), \\ \phi(0,x) = 0 \quad (x \in \mathbb{R}), \\ \lim\limits_{x \to \pm\infty} \phi(t,x) = 0 \quad \big(t \geq 0 \big), \end{cases} \tag{3.2}$$

where $\mu > 0,\ \delta \in \mathbb{R}$. Then we shall state the corresponding theorem for ϕ we should prove.

Theorem 3.1. *Let $\mu > 0,\ \delta,\ u_{\pm} \in \mathbb{R}$. There exists the unique global in time solution ϕ of the Cauchy problem* (3.2) *satisfying*

$$\begin{cases} \phi \in C^0\big(\, [0,\, \infty)\, ; H^1 \,\big), \\ \partial_x \phi \in C^0\big(\, [0,\, \infty)\, ; L^2 \,\big) \cap L^2(\, 0,\, \infty\, ; H^1\,), \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} |\,\phi(t,x)\,| = 0.$$

Proof of Theorem 3.1. Because the uniquely existence of the global in time solution ϕ to (3.2) is standard, we omit the proof (cf. [1-4], [6-26]). Therefore we only show the asymptotic behavior

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} |\,\phi(t,x)\,| = 0. \tag{3.3}$$

We first note by applying the energy method that there exists a positive constant $C_{\mu,\delta}$ not depending on t such that

$$\|\,\phi(t)\,\|_{H^1}^2 + \int_0^t \|\,\partial_x\phi(\tau)\,\|_{H^1}^2\,\mathrm{d}\tau \le C_{\mu,\delta}\int_0^t \|\,\partial_x^2 U^{\mathrm{V,C}}(\tau)\,\|_{H^1}^2\,\mathrm{d}\tau \quad \big(\,t\ge 0\,\big). \tag{3.4}$$

By using Lemma 2.1, we easily see

$$\int_0^t \|\,\partial_x^2 U^{\mathrm{V,C}}(\tau)\,\|_{H^1}^2\,\mathrm{d}\tau \le C_{\mu,\delta,u_\pm} \quad \big(\,t\ge 0\,\big). \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain the following uniform estimate.

$$\sup_{t\ge 0}\|\,\phi(t)\,\|_{H^1}^2 + \int_0^\infty \|\,\partial_x\phi(t)\,\|_{H^1}^2\,\mathrm{d}t < \infty. \tag{3.6}$$

From (3.6), we can get

$$\int_0^\infty \left|\,\frac{\mathrm{d}}{\mathrm{d}t}\|\,\partial_x\phi(t)\,\|_{L^2}^2\,\right|\mathrm{d}t < \infty. \tag{3.7}$$

Therefore, by using (3.6) and (3.7), we have the L^2-stability as follows.

$$\|\,\partial_x\phi(t)\,\|_{L^2} \to 0 \quad (t\to\infty). \tag{3.8}$$

By the Sobolev inequality, we obtain from (3.8) that the desired asymptotic behavior, that is,

$$\sup_{x\in\mathbb{R}}|\,\phi(t,x)\,| \le \sqrt{2}\,\|\,\phi(t)\,\|_{L^2}^{\frac12}\|\,\partial_x\phi(t)\,\|_{L^2}^{\frac12} \to 0 \quad (t\to\infty). \tag{3.9}$$

Thus, the proof of Theorem 3.1 is completed.

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